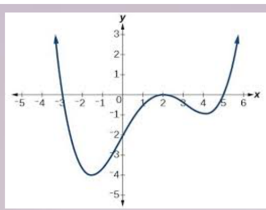

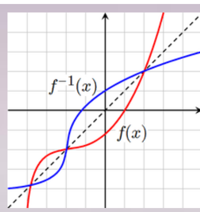

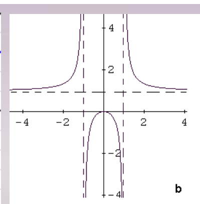

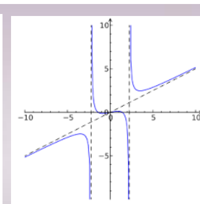

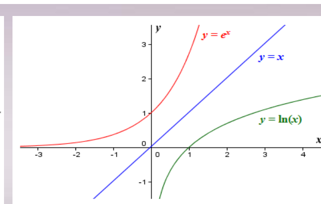

Math 1050 ~ College Algebra

23 Systems of Linear Equations: Augmented Matrices

$$\begin{aligned} -3x + 4y &= 5 \\ 2x - y &= -10 \end{aligned}$$

$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

Learning Objectives

- Write a system of linear equations as an augmented matrix.
- Perform row operations on a matrix.
- Convert an augmented matrix to row echelon form.
- Convert an augmented matrix to reduced row echelon form.
- Use matrix row operations to solve systems of linear equations.

Solving three equations in three variables takes a bit of writing. We can reduce some of that by writing the set of equations as an **augmented matrix**. A matrix is a structured array of numbers grouped by square brackets. We only write the coefficients and constants.

Ex 1a: Write this set of equations as an augmented matrix.

$$
\begin{aligned}
x - y + z &= 4 \\
x + 3y - 2z &= -3 \\
3x + 2y + 2z &= 6
\end{aligned}
\qquad
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 1 & 3 & -2 & -3 \\ 3 & 2 & 2 & 6 \end{array}\right]
$$

We can operate on this matrix using the same set of rules that we used for Gaussian elimination in the last section.

Sometimes called row operations

1. Exchange two rows.

2. Multiply a row by a nonzero constant.

3. Temporarily multiply a row by a nonzero constant and add it to another row, replacing either of those rows with the result.

Ex 1b: Solve this system of equations from 1a by using these options.

$$
\begin{array}{c}
\begin{array}{cccc} -1 & 1 & -1 & -4 \end{array} \\
(-1)\left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 1 & 3 & -2 & -3 \\ 3 & 2 & 2 & 6 \end{array}\right]
\end{array}
\quad
\begin{array}{c}
\begin{array}{cccc} -3 & 3 & -3 & -12 \end{array} \\
(-3)\left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 4 & -3 & -7 \\ 3 & 2 & 2 & 6 \end{array}\right]
\end{array}
\quad
\begin{array}{c}
(3)\left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 4 & -3 & -7 \\ 0 & 5 & -1 & 6 \end{array}\right] \\
\begin{array}{cccc} 0 & -15 & 3 & 18 \end{array}
\end{array}
$$

$$
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & -11 & 0 & 11 \\ 0 & 5 & -1 & -6 \end{array}\right]
$$

middle row: $-11y = 11$

$\boxed{y = -1}$

bottom row:

$5y - z = -6$

$5(-1) - z = -6$

$-z = -1$

$z = 1$

top row:

$x - y + z = 4$

$x - (-1) + 1 = 4$

$x = 2$

Solution: $\boxed{(2, -1, 1)}$

Row Echelon Form

Ex 2: Solve for x, y and z in these augmented matrices representing a set of equations.

a) Reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$x = -3$
$y = 1$
$z = 0$

solution $\boxed{(-3, 1, 0)}$

b) row echelon form

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{l} x - 2y + 3z = 9 \quad (1) \\ y + 3z = 5 \quad (2) \\ z = 2 \end{array}$$

(2) $y + 3(2) = 5 \iff y = -1$

(1) $x - 2(-1) + 3(2) = 9 \iff x = 1$

Solution: $\boxed{(1, -1, 2)}$

These are examples of row echelon form and reduced row echelon form. Our goal is to reduce a set of equations to one of these forms, using Gaussian elimination.

Ex 3: Solve this set of equations by reducing them to each of the forms above.

$$\begin{aligned} 2x - 2y + z &= -9 \\ x + y + 2z &= -5 \\ x \quad\quad - z &= 11 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -9 \\ 1 & 1 & 2 & -5 \\ 1 & 0 & -1 & 11 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{cccc} -1 & 0 & 1 & -11 \end{array} \\ (-1)\left[\begin{array}{ccc|c} 1 & 0 & -1 & 11 \\ 1 & 1 & 2 & -5 \\ 2 & -2 & 1 & -9 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} -2 & 0 & 2 & -22 \end{array} \\ (-2)\left[\begin{array}{ccc|c} 1 & 0 & -1 & 11 \\ 0 & 1 & 3 & -16 \\ 2 & -2 & 1 & -9 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} 0 & 2 & 6 & -32 \end{array} \\ (2)\left[\begin{array}{ccc|c} 1 & 0 & -1 & 11 \\ 0 & 1 & 3 & -16 \\ 0 & -2 & 3 & -31 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 11 \\ 0 & 1 & 3 & -16 \\ (\tfrac{1}{9})\; 0 & 0 & 9 & -63 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 11 \\ 0 & 1 & 3 & -16 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

this is now in row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 3 & -16 \\ (-3)\; 0 & 0 & 1 & -7 \end{array}\right]$$

$0 \quad 0 \quad -3 \quad 21$

this is reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

$x = 4$
$y = 5$
$z = -7$

Solution: $\boxed{(4, 5, -7)}$

Ex 4: Set up this set of equations as an augmented matrix and put in reduced row echelon form.

$A = 2$

$B = -1$

$3A + 2C = 0$

$2B - D = -5$

$$\begin{array}{cc} & \begin{array}{ccccc} -3 & 0 & 0 & 0 & -6 \end{array} \\ (-3) & \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 3 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -1 & -5 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccccc} 0 & -2 & 0 & 0 & 2 \end{array} \\ & \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -6 \\ 0 & 2 & 0 & -1 & -5 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ \\ (\tfrac{1}{2}) \\ (-1) \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -6 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

this is now in reduced row echelon form

$A = 2 \quad C = -3$

$B = -1 \quad D = 3$

$\boxed{(2, -1, -3, 3)}$

What happens if the system has infinite solutions or no solutions?

Ex 5: Write the solution to each system below.

a) $\left[\begin{array}{ccc:c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\boxed{(-3, 1, 0)}$

b) $\left[\begin{array}{ccc:c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$

bottom row:

$0 = 2$

false statement

N.S.

c) $\left[\begin{array}{ccc:c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

bottom row:

$0 = 0$

true statement

infinitely many solutions

finish (c): $\left[\begin{array}{ccc:c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

middle row: $y = 1$

top row: $x = -3$

line of intersection:

$\boxed{(-3, 1, z)}$