Common and Natural Logarithms

Base 10 is commonly used in logarithms. Thus, when no base is indicated, it is assumed to be base 10.

\[ \log x = \log_{10} x \]

Ex 1: Evaluate these.

a) \( \log 1,000,000 \)  
 b) \( \log (10^3) \)  
 c) \( \log 0.01 \)  
 d) \( \log (\text{a trillion}) \)

Another base is the irrational number, \( e \), called the natural base. This is written using \( \ln x \).

\[ \log_e x = \ln x \]

Ex 2: Evaluate these.

a) \( \ln e \)  
 b) \( \ln e^3 \)  
 c) \( \ln e^8 \)  
 d) \( \ln \left(\frac{1}{e^2}\right) \)
Natural Exponential Base

\[ e \approx 2.718281828459 \ldots \]

Ex 4: Sketch a graph of \( y = e^x \).

The exponential base is used in financial and scientific calculations which we will explore in a later chapter.

Logarithm Properties

Let \( b \) be a positive number, not equal to 1, and let \( x \) be a positive number. 

- \( \log_b 1 = 0 \)
- \( \log_b b = 1 \)
- \( \log_b b^x = x \)
- If \( \log_b x = \log_b y \), then \( x = y \)

Ex 5: Evaluate these.

a) \( \ln 1 \) \quad b) \( \log 100 \) \quad c) \( \ln e^\pi \) \quad d) \( \log (10^{0.2}) \)

Ex 6: Determine the value of \( x \) for each of these.

a) \( \log x = \log (y + 5) \) \quad b) \( \ln x = \ln (\pi + 1) \)
Properties of Logarithms

Change of Base Property
Let \( a \) and \( b \) be positive numbers, not equal to 1, and let \( x \) be a positive number.

\[
\log_b x = \frac{\log_a x}{\log_a b}
\]

Ex 7: True or false? \( \log_3 8 = \frac{\log_2 8}{\log_2 3} \)

Ex 8: Use your calculator to give an approximate value for these.

\begin{align*}
\text{a)} & \quad \log_5 3 \quad \text{b)} \quad \log_5 50 \quad \text{c)} \ln 8 \quad \text{d)} \log_6 0.0002
\end{align*}

Inverse Properties
Let \( b \) be a positive number, not equal to 1.

\[
b^{\log_b x} = x, \text{ for any positive number } x
\]

\[
\log_b b^x = x, \text{ for any real number } x
\]

Ex 9: Use the inverse properties to simplify.

\begin{align*}
\text{a)} & \quad \ln e - 2 \quad \text{b)} \quad \log_3 1 \quad \text{c)} \quad 6^{\log_6 20} \quad \text{d)} \quad \log_5 3^{10}
\end{align*}

Exponential Sum and Difference Properties of Logarithms
Let \( b \) be a positive number, not equal to 1, and let \( u \) and \( v \) be positive numbers.

\[
\log_b (uv) = \log_b u + \log_b v
\]

\[
\log_b \left( \frac{u}{v} \right) = \log_b u - \log_b v
\]

\[
\log_b u^m = m \log_b u
\]

Ex 10: Use these properties to expand these expressions.

\begin{align*}
\text{a)} & \quad \log \sqrt{x^2(x + 2)} \quad \text{b)} \quad \ln \left( \frac{x^2 - 1}{x^3} \right), \quad x > 1
\end{align*}

Ex 11: Use these properties to contract these expressions into a single term.

\begin{align*}
\text{a)} & \quad 3 \log x + 4 \log y - 5 \log z \quad \text{b)} \quad \frac{1}{2} [\ln(x + 1) + 2 \ln(x - 1)] - 6 \ln x
\end{align*}