Math 1050 ~ College Algebra

12 Introduction to Rational Functions

Learning Objectives

- Identify a rational function.
- Determine the domain of a rational function.
- Find the x- and y-intercepts for a rational function.
- Identify vertical and horizontal asymptotes.
- Graph irreducible rational functions with constant or first degree numerators and denominators of degree one.
A rational function is a ratio of two polynomial functions.

\[ f(x) = \frac{N(x)}{D(x)} \]

where \( N(x) \) and \( D(x) \) are polynomials.

Note: all polynomials are a subset of rational functions.

\( N(x) = \) numerator polynomial
\( D(x) = \) denominator polynomial

Ex 1: Determine which of these functions are rational functions.

a) \( f(x) = \frac{x^2+1}{x+4} \quad \text{both } N(x) \text{ and } D(x) \text{ are polynomials} \)

b) \( f(x) = \frac{3x+2}{\sqrt{x} - 3} \quad D(x) \text{ is NOT a polynomial} \)

c) \( f(x) = \frac{x^2-2x-3}{\pi} \quad N(x) \text{ is 2nd degree polynomial, } D(x) \text{ is a 0-degree polynomial} \)

d) \( f(x) = \frac{x^2+5}{x^2-25} \quad N(x) \text{ is NOT a polynomial} \)

Vertical Asymptotes of Simplified Rational Functions

- determined by finding disallowed denominator values
- line \( x = a \) where \( D(a) = 0 \)
- graph will never cross or touch

\( i.e. x\)-values that make \( D(x) = 0 \)

Ex 2: Find the domain and the vertical asymptotes for these functions.

a) \( f(x) = \frac{2x^2}{x^2-1} \)

\( f(x) = \frac{2x^2}{(x-1)(x+1)} \)

\( x \neq 1, -1 \) domain

VA: \( x = 1 \) and \( x = -1 \)

domain: \( x \in \mathbb{R} \) (or \( (-\infty, \infty) \))

there are no VA.

b) \( f(x) = \frac{3x}{x^2+1} \)

domain: \( x \neq 0, 2 \)

(\( -\infty, 0 \)) \cup (0, 2) \cup (2, \infty)

VA: \( x = 0, x = 2 \)

c) \( f(x) = \frac{x+4}{4x-2x^2} \)

\( x+4 \)

\( 2x(2-x) \)
Horizontal Asymptotes
- end behavior of the graph
- line $y = b$ where $\lim_{x \to \infty} f(x) = b$
- graph may cross it
- depends on degree of $N(x)$ and $D(x)$
  - degree $(N(x)) < \text{degree (} D(x))$, $y = 0$
  - degree $(N(x)) = \text{degree (} D(x))$, $y = \text{ratio of the leading coefficients.}$

Ex 3: Find the horizontal asymptotes of these functions.

a) $f(x) = \frac{2x^2}{x^2 - 1}$

as $x$ gets really huge, what matters most is
$$\frac{2x^2}{x^2} = 2$$
$\Rightarrow$ HA: $y = 2$

b) $f(x) = \frac{3x}{x^2 + 1}$

as $x$ gets really huge, this behaves similar to
$$\frac{3x}{x^2} = \frac{3}{x}$$
(note: thinking is that $\frac{3}{\text{super huge}}$ = super small)
$\Rightarrow$ HA: $y = 0$

c) $f(x) = \frac{x + 4}{4x - 2x^2}$

as $x$ gets super huge, $f(x)$ will eventually behave like
$$\frac{\frac{1}{x}}{-2x} = \frac{1}{-2x} \rightarrow 0$$

HA: $y = 0$
Ex 4: For each of these functions, determine the x and y-intercepts, vertical and horizontal asymptotes and draw a quick sketch.

a) \( f(x) = \frac{2x^2}{x^2 - 1} \)

   Vertical Asymptote (VA): \( x = 1, x = -1 \)
   Horizontal Asymptote (HA): \( y = 2 \)
   (Note: this is an even function)

   x-intercept: \( \frac{2x^2}{x^2 - 1} = 0 \)
   \( 2x^2 = 0 \)
   \( x = 0 \)
   y-intercept: \( f(0) = \frac{0}{0-1} = 0 \)
   (0,0)

b) \( f(x) = \frac{3x}{x^2 + 1} \)

   Vertical Asymptote (VA): none
   Horizontal Asymptote (HA): \( y = 0 \)
   (Note: this is an odd function)

   x-intercept: \( \frac{3x}{x^2 + 1} = 0 \)
   \( 3x = 0 \)
   \( x = 0 \)
   y-intercept: \( f(0) = \frac{0}{0+1} = 0 \)
   (0,0)

   No y-intercept.


c) \( f(x) = \frac{x+4}{4x-2x^2} \)

   Vertical Asymptote (VA): \( x = 0, x = -2 \)
   Horizontal Asymptote (HA): \( y = 0 \)

   x-intercept: \( \frac{x+4}{4x-2x^2} = 0 \)
   \( x+4 = 0 \)
   \( x = -4 \)
   y-intercept: \( f(0) = \frac{4}{-2} = -2 \)
   (0,-2)

   x + y = 0
   \( x = -4 \)
   \( (-4,0) \)