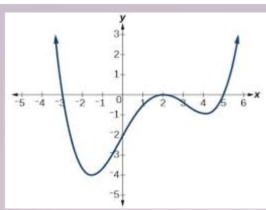
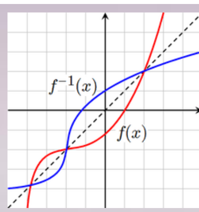
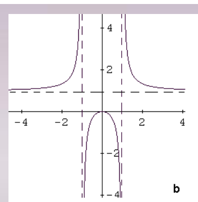
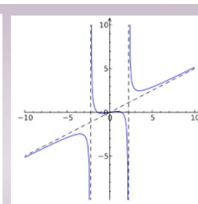
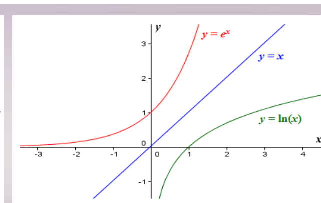









# Math 1050 ~ College Algebra

$$-3x + 4y = 5$$
$$2x - y = -10$$
$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$
$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

## 10 Complex Zeros of Polynomials

### Learning Objectives

- Perform operations on complex numbers.
- Find all complex zeros of a polynomial.
- Factor a polynomial to linear and irreducible quadratic factors.
- Use the conjugate of a complex zero to identify an additional zero.
- Create a polynomial given information that includes complex zeros.

**Complex Numbers** $\mathbb{C}$

The <u>imaginary unit *i*</u> satisfies the following properties:

- $i^2 = -1$
- If $c$ is a real number, $c \geq 0$ then $\sqrt{-c} = (\sqrt{c}) \cdot i$

ex $\sqrt{-5} = i\sqrt{5}$

A <u>complex number</u> is a number of the form $\boxed{a + bi}$ where $a$ and $b$ are real numbers and $i$ is the imaginary unit.

The real numbers are a subset of the complex numbers. $\mathbb{R} \subseteq \mathbb{C}$

subset

The <u>conjugate</u> of a complex number, $a + bi$ is $a - bi$.

Expressed in symbols, $\overline{a+bi} = a - bi$ .

ex $3+4i$ conjugate is $3-4i$

Ex 1: Identify *a, b* and the conjugate of each of these complex numbers.

a) $-2 + 5i$

$a = -2$

$b = 5$

conjugate

$-2-5i$

b) $6i$ $= 0+6i$

$a = 0$

$b = 6$

conjugate

$0-6i = -6i$

c) $53$

$= 53+0i$

$a = 53$

$b = 0$

conjugate

$53-0i$

$= 53$

d) $\pi - i$

$a = \pi$

$b = -1$

conjugate

$\pi + i$

Arithmetic on these numbers is as expected.

EX 2: Perform these operations on complex numbers.

a) $(1-3i)+(2+5i)$

$= (1+2)+(-3+5)i$

$= \boxed{3+2i}$

b) $(1-3i)(2+5i)$

$= 2+5i-6i-15i^2$

$= 2-i-15(-1)$

$= 2-i+15$

$= \boxed{17-i}$

c) $(1-3i)-(2+5i)$

$= (1-2)+(-3-5)i$

$= \boxed{-1-8i}$

d) $\dfrac{1-3i}{2+5i}$

$= \left(\dfrac{1-3i}{2+5i}\right)\left(\dfrac{2-5i}{2-5i}\right)$

$= \dfrac{2-5i-6i+15i^2}{4-\cancel{10i}+\cancel{10i}-25i^2}$

$= \dfrac{2-11i+15(-1)}{4-25(-1)}$

$= \dfrac{-13-11i}{29} = \boxed{\dfrac{-13}{29}+\dfrac{-11}{29}i}$

e) $\sqrt{-3}\sqrt{-12}$

$= (i\sqrt{3})(i\sqrt{12})$

$= i^2\sqrt{36}$

$= 6(-1)$

$= \boxed{-6}$

f) $\sqrt{(-3)(-12)}$

$= \sqrt{36}$

$= \boxed{6}$

Ex 3: Perform this multiplication.

$(x-(1+2i))(x-(1-2i))$

$= x^2-x(1-2i)-(1+2i)x$
$+(1+2i)(1-2i)$

$= x^2-x+\cancel{2ix}-x-\cancel{2ix}+1-\cancel{2i}+\cancel{2i}-4i^2$

$= x^2-2x+1-4(-1) = \boxed{x^2-2x+5}$

(note: this is a variable expression, not a numerical expression like example above)

$(1+2i)$ and $(1-2i)$ are conjugates.

**Complex Roots of Polynomial Functions** (we're assuming coefficients in polynomial are all real)

The Fundamental Theorem of Algebra and Complex Factorization.

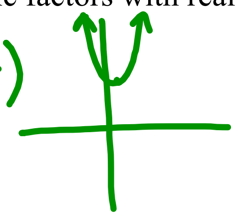
If $f$ is a polynomial function with degree $n \geq 1$:

- $f$ has at least one complex zero.
- In actuality, $f$ has exactly $n$ zeros, counting multiplicities.
- $f$ has precisely $n$ factors.

Furthermore:

- Complex zeros occur in conjugate pairs.
- Every polynomial can be factored into linear and quadratic factors with real coefficients.

vertex: $x = \frac{2}{2(3)} = \frac{1}{3}$ vertex$\left(\frac{1}{3}, \frac{5}{3}\right)$



Ex 4: Determine the complex zeros of $f(x) = 3x^2 - 2x + 2$ .

$0 = 3x^2 - 2x + 2$

$$x = \frac{2 \pm \sqrt{4-4(3)(2)}}{2(3)} = \frac{2 \pm \sqrt{4(1-6)}}{6} = \frac{2 \pm 2\sqrt{-5}}{6}$$

$$= \frac{2(1 \pm \sqrt{5}i)}{6} = \frac{1 \pm \sqrt{5}i}{3}$$

two zeros: $\boxed{x = \frac{1}{3} + \frac{\sqrt{5}}{3}i, \frac{1}{3} - \frac{\sqrt{5}}{3}i}$

Ex 5: Given $x + 3i$ is a factor of $f(x) = 2x^3 - 11x^2 + 18x - 99$, find all other zeros.

$n = 3 \Rightarrow$ there are 3 zeros.

if $x + 3i$ is a factor, then $x + 3i = 0$

$x = \boxed{-3i}$ is a zero of $f(x)$.

$\Rightarrow x = \boxed{3i}$ must also be a zero of $f(x)$ (since complex zeros come as conjugate pairs).

$\Rightarrow (x - 3i)$ is a factor of $f(x)$.

$$(x+3i)(x-3i) = x^2 - \cancel{3ix} + \cancel{3ix} - 9i^2 = x^2 - 9(-1) = x^2 + 9$$

$$\begin{array}{r} 2x - 11 \\ x^2+9 \overline{) 2\cancel{x}^3 - 11x^2 + 1\cancel{8x} - 99} \\ -(2\cancel{x}^3 \qquad + \cancel{1}8x) \\ \hline -11x^2 - 99 \\ -(-11x^2 - 99) \\ \hline 0 \end{array}$$

$\Rightarrow f(x) = (x+3i)(x-3i)(2x-11)$

$2x - 11 = 0$

$2x = 11$

$x = \boxed{\frac{11}{2}}$

$\boxed{\text{3 zeros: } 3i, -3i, \frac{11}{2}}$

Ex 6: Use the techniques in this section and the last to find all the zeros of $f(x) = x^5 + 6x^4 + 10x^3 + 6x^2 + 9x$. $= x(x^4 + 6x^3 + 10x^2 + 6x + 9)$

$g(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

zeros:
$x = 0$
$x = -3$ (mult. 2)
$x = i$
$x = -i$

possible rational roots/zeros:
$\pm 1, \pm 3, \pm 9$

using Descartes Rule of Signs:
$\Rightarrow$ 0 positive roots
$\Rightarrow$ check possible roots/zeros
$-1, -3, -9$

| | | | | | |
|---|---|---|---|---|---|
| $-1$ | 1 | 6 | 10 | 6 | 9 |
| | | $-1$ | $-5$ | $-5$ | $-1$ |
| | 1 | 5 | 5 | 1 | 8 |

remainder

$\Rightarrow -1 = x$ is NOT a root/zero of $g(x)$

| | | | | | |
|---|---|---|---|---|---|
| $-3$ | 1 | 6 | 10 | 6 | 9 |
| | | $-3$ | $-9$ | $-3$ | $-9$ |
| | 1 | 3 | 1 | 3 | 0 |

remainder

$\Rightarrow x = -3$ is a root/zero of $g(x)$

$\Rightarrow g(x) = (x+3)(x^3 + 3x^2 + x + 3)$

| | | | | |
|---|---|---|---|---|
| $-3$ | 1 | 3 | 1 | 3 |
| | | $-3$ | 0 | $-3$ |
| | 1 | 0 | 1 | 0 |

remainder

$\Rightarrow x = -3$ is a root/zero again.

$\Rightarrow g(x) = (x+3)^2(x^2+1)$

$$\Rightarrow g(x) = x^4 + 6x^3 + 10x^2 + 6x + 9 = \underbrace{(x+3)^2}_{\text{linear factor}}\underbrace{(x^2+1)}_{\text{irreducible quadratic factor}}$$

(if we only allow real roots)

$g(x) = (x+3)^2(x^2+1)$

$g(x) = (x+3)^2(x-i)(x+i)$

zeros/roots of $g(x)$ are $-3, i, -i$

multiplicity 2

each has multiplicity 1

Note: think about $x^2+1$ as $x^2-(-1)$ and now it's a difference of squares.

Note: $i$ and $-i$ are complex conjugates.