Math 1050 ~ College Algebra

10 Complex Zeros of Polynomials

Learning Objectives

- Perform operations on complex numbers.
- Find all complex zeros of a polynomial.
- Factor a polynomial to linear and irreducible quadratic factors.
- Use the conjugate of a complex zero to identify an additional zero.
- Create a polynomial given information that includes complex zeros.
Complex Numbers \( \mathbb{C} \)

The imaginary unit \( i \) satisfies the following properties:

- \( i^2 = -1 \)
- If \( c \) is a real number, \( c \geq 0 \) then \( \sqrt{-c} = (\sqrt{c}) \cdot i \)

\[ \alpha \sqrt{-5} = i \sqrt{5} \]

A complex number is a number of the form \( a + bi \) where \( a \) and \( b \) are real numbers and \( i \) is the imaginary unit.

The real numbers are a subset of the complex numbers.

The conjugate of a complex number, \( a + bi \) is \( a - bi \).

Expressed in symbols, \( a + bi = a - bi \).

\[ \approx 3 + 4i \text{ conjugate is } 3 - 4i \]

Ex 1: Identify \( a, b \) and the conjugate of each of these complex numbers.

a) \( -2 + 5i \)
\[
\begin{align*}
a &= -2 \\
b &= 5 \\
\text{Conjugate} &= -2 - 5i
\end{align*}
\]

b) \( 6i \)
\[
\begin{align*}
a &= 0 \\
b &= 6 \\
\text{Conjugate} &= 0 - 6i
\end{align*}
\]

c) \( 53 \)
\[
\begin{align*}
a &= 53 \\
b &= 0 \\
\text{Conjugate} &= 53 - 0i = 53
\end{align*}
\]

d) \( \pi - i \)
\[
\begin{align*}
a &= \pi \\
b &= -1 \\
\text{Conjugate} &= \pi + i
\end{align*}
\]
Arithmetic on these numbers is as expected.

EX 2: Perform these operations on complex numbers.

a) \((1-3i)+(2+5i)\)
   \[\begin{align*}
   &= (1+2) + (-3+5)i \\
   &= 3 + 2i
   \end{align*}\]

b) \((1-3i)(2+5i)\)
   \[\begin{align*}
   &= 2+5i-6i-15i^2 \\
   &= 2-i-15(-1) \\
   &= 2-i+15 \\
   &= 17-i
   \end{align*}\]

c) \((1-3i)-(2+5i)\)
   \[\begin{align*}
   &= (1-2) + (-3-5)i \\
   &= -1-8i
   \end{align*}\]

d) \(\frac{1-3i}{2+5i}\)
   \[\begin{align*}
   &= (\frac{1-3i}{2+5i})(\frac{2-5i}{2-5i}) \\
   &= \frac{2-5i-6i+15i^2}{4-10i+10i-25i^2} \\
   &= \frac{2-11i+15(-1)}{4-25(-1)} \\
   &= \frac{-13-11i}{29} \quad \text{(note: this is a variable expression, not a numerical expression like example above)}
   \end{align*}\]

Ex 3: Perform this multiplication.

\((x-(1+2i))(x-(1-2i))\)
   \[\begin{align*}
   &= x^2 - x(1-2i) - (1+2i)x \\
   &\quad + (1+2i)(1-2i) \\
   &= x^2 - x + 2x - x + 1 - 2i + 2i - 4i^2 \\
   &= x^2 - 2x + 1 - 4(-1) \\
   &= x^2 - 2x + 5
   \end{align*}\]
**Complex Roots of Polynomial Functions**

(we're assuming coefficients in polynomial are all real)

The Fundamental Theorem of Algebra and Complex Factorization.

If \( f \) is a polynomial function with degree \( n \geq 1 \):
- \( f \) has at least one complex zero.
- In actuality, \( f \) has exactly \( n \) zeros, counting multiplicities.
- \( f \) has precisely \( n \) factors.

Furthermore:
- Complex zeros occur in conjugate pairs.
- Every polynomial can be factored into linear and quadratic factors with real coefficients.

Ex 4: Determine the complex zeros of \( f(x) = 3x^2 - 2x + 2 \).

\[
x = \frac{2 \pm \sqrt{4 - 4(3)(2)}}{2(3)} = \frac{2 \pm \sqrt{4(1 - 6)}}{6} = \frac{2 \pm 2\sqrt{5}}{6}
\]

\[
x = \frac{2(1 \pm \sqrt{5}i)}{6} = \frac{1 \pm \sqrt{5}i}{3}
\]

two zeros: \( x = \frac{1}{3} + \frac{\sqrt{5}}{3}i, \frac{1}{3} - \frac{\sqrt{5}}{3}i \)

Ex 5: Given \( x + 3i \) is a factor of \( f(x) = 2x^3 - 11x^2 + 18x - 99 \), find all other zeros.

if \( x + 3i \) is a factor, then \( x + 3i = 0 \) \( \Rightarrow x = -3i \) is a zero of \( f(x) \).

\( \Rightarrow x = -3i \) must also be a zero of \( f(x) \) (since complex zeros come as conjugate pairs).

\( \Rightarrow (x + 3i) \) is a factor of \( f(x) \).

\[
(x + 3i)(x - 3i) = x^2 - 3i^2x + 3ix - 9i^2 = x^2 - 9(-1) = x^2 + 9
\]

\[
x^2 + 9\sqrt{2x - 11} - 11x^2 + 18i - 99 = 0, 2x - 11 = 0, 2x = 11, x = \frac{11}{2}
\]

\[
\frac{-11x^2 - 99}{(-11x^2 - 99)} \Rightarrow f(x) = (x + 3i)(x - 3i)(2x - 11)
\]

3 zeros: \( \frac{3}{2}, -3i, \frac{11}{2} \)
Ex 6: Use the techniques in this section and the last to find all the zeros of 
\( f(x) = x^5 + 6x^4 + 10x^3 + 6x^2 + 9x \).

\[ g(x) = x^4 + 6x^3 + 10x^2 + 6x + 9 \]

possible rational roots/zeros: 
\( \pm 1, \pm 3, \pm 9 \)

using Descartes Rule of Signs:
\( \Rightarrow 0 \) positive roots

\( \Rightarrow \) check possible roots/zeros
-1, -3, -9

\[ \begin{array}{c|ccccc}
-1 & 1 & 6 & 10 & 6 & 9 \\
 & -1 & -5 & -5 & -1 \\
\hline
 & 1 & 5 & 5 & 1 & 8
\end{array} \]

remainder \( \Rightarrow -1 \)x is NOT a root/zero of \( g(x) \)

\[ \begin{array}{c|ccccc}
-3 & 1 & 6 & 10 & 6 & 9 \\
 & -3 & -9 & -3 & -9 \\
\hline
 & 1 & 3 & 1 & 3 & 0
\end{array} \]

remainder \( \Rightarrow x=-3 \) is a root/zero of \( g(x) \)

\( \Rightarrow g(x) = (x+3)(x^3+3x^2+x+3) \)

\( \Rightarrow x=-3 \) is a root/zero again.

\( \Rightarrow g(x) = (x+3)^2(x^3+1) \)

\[ g(x) = (x+3)^2(x^3+1) \]

linear irreducible factor quadratic factor

if we only allow real roots

\[ g(x) = (x+3)^2(x^3+1) \]

Note: think about \( x^3+1 \) as \( x^3-(-i)^3 \) and now it's a difference of squares.

zeros/roots of \( g(x) \)

are \(-3, i, -i\)

\( i \) and \(-i\) are complex conjugates.

Note: each has multiplicity 1