9.5 The Binomial Theorem

* Use the Binomial Theorem to calculate binomial coefficients.

* Use Pascal's Triangle to calculate binomial coefficients.

* Find the $n$th term in a binomial expansion.

\[
(3x - 2y)^6 
\]

\[
(2x + 5)^8 \rightarrow 6^{th} \text{ term}
\]
What does the word **binomial** mean?

\[
\begin{align*}
3x-2 & \quad x^2+y & \quad x-2y \\
(a+b)^0 &= 1 \\
(a+b)^1 &= a + b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)(a^2+2ab+b^2) &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\
(a+b)^4 &= a^3 + 3a^2b + 3ab^2 + b^3
\end{align*}
\]
What does 7! mean?  

\[ 7! \text{ Factorial} \rightarrow 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \]

Example 1: Determine the value of each of these.

a) 4!  
\[ = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \]

b) 10!  
\[ = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3,628,800 \]

c) 12!/10!  
\[ = \frac{12 \cdot 11 \cdot 10!}{10!} = 132 \]

d) n!  
\[ = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1 \]

e) (n+2)!  
\[ = (n+2)(n+1) \ldots 3 \cdot 2 \cdot 1 \]

f) 0!  
\[ = 1 \text{ By def.} \]
Example 2: A pizza shop offers 4 different toppings, Onions, Mushrooms, Pepperoni and Ham. How many 'different' pizzas can you order having none, one, two, three or all four toppings?

\[
\begin{align*}
\{ & \emptyset \\
\{& o \} \ 
\{& e \} \\
\{& o, m \} \\
\{& o, p \} \\
\{& o, m, p \} \\
\{& o, h \} \\
\{& o, m, h \} \\
\{& o, p, h \} \\
\{& o, m, p, h \}
\end{align*}
\]

1 4 6 4 1

Combination of \( n \) things taken \( r \) at a time.

What does \( \binom{n}{r} \) mean? I have \( n \) things, I choose \( r \) of them.

\[
\binom{n}{r} = \frac{n!}{(n-r)!r!} = \binom{n}{r} = \binom{n}{r}
\]

Determine the value of each of these.

\[
\begin{align*}
\binom{4}{0} & = \frac{4!}{0!4!} = 1 \\
\binom{4}{1} & = \frac{4!}{3!1!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 4 \\
\binom{4}{2} & = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 2} = 6 \\
\binom{4}{3} & = \frac{4!}{1!3!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 3 \cdot 2 \cdot 1} = 4 \\
\binom{4}{4} & = \frac{4!}{0!4!} = 1
\end{align*}
\]
Example 3: Determine the value of each of these and make up a question it might answer.

\[ \binom{6}{2} = \frac{6!}{4! \cdot 2!} = \frac{6 \cdot 5 \cdot 4!}{4! \cdot 2} = 15 \]

How many ways can I select 2 friends out of 6 to take to dinner?

\[ \binom{12}{10} = \frac{12!}{2! \cdot 10!} = \frac{12 \cdot 11 \cdot 10!}{2 \cdot 10!} = 66 \]

10 of 12 books to read this summer?

\[ \binom{7}{4} = \frac{7!}{3! \cdot 4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3 \cdot 2 \cdot 1 \cdot 4!} = 35 \]

4 of 7 coins to give away.

\[ \binom{15}{0} = \frac{15!}{15! \cdot 0!} = 1 \]

0 out of 15 cards to give away.
**Binomial Theorem** and Pascal's Triangle

\[(x+y)^n = \sum_{j=0}^{\infty} \binom{n}{j} x^{n-j} y^j\]

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\[\binom{n}{0} \binom{n}{1} \binom{n}{2} \binom{n}{3} \binom{n}{4} \binom{n}{5} \]

So, \((a+b)^5 = 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5\]

\[a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\]
Example 4: Expand this binomial. \((2x - y)^4 =\)

\[
(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4
\]

\(a = 2x\) \quad b = -y \checkmark

\[
(2x)^4 + 4(2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4
\]

\[
16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4
\]

\(\checkmark\)
Pascal's Triangle

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Example 5: How do we find the $x$th term in the expansion of $(2x-y)^{10}$ without writing the entire expansion?

\[
\binom{10}{4} (2x)^6 (-y)^4 = \frac{10!}{4!6!} (2x)^6 (-y)^4 \\
210 \cdot 64x^6y^4 = 13,440x^6y^4
\]
Example 6: An interesting application of Pascal's Triangle is in probability. In a family of six children, what is the probability that two are boys and the rest are girls?

\[
P(2 \text{ boys in 6 children}) = \frac{\binom{6}{2}}{2^6} = \frac{15}{64} \approx 0.234 \approx 23\% 
\]