9.3 Geometric Sequences and Series

In sections 9.3 you will learn to:

- Recognize, write and find the nth terms of geometric sequences.
- Find the nth partial sums of geometric sequences.
- Find the sums of infinite geometric sequences.
- Use geometric sequences to model and solve real-life problems.

$1, 2, 4, 8, \ldots, 2^{n-1}, \ldots$

$\sum_{i=1}^{3} 2^i = 7 \qquad \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$

applications

A sequence $a_1, a_2, a_3, \ldots, a_n$ is said to be geometric is the ratio between consecutive terms remains constant.

$\frac{a_k}{a_{k-1}} = r$ constant

Which of these are geometric sequences?

$\frac{1}{3} \quad \frac{1/3}{1} = \frac{1}{3} \quad \frac{1/9}{1/3} = \frac{1}{3}$

3, 1, 1/3, 1/9, ... $r = \frac{1}{3}$ geom.

✓ 2, 4, 8, 16, ... geom. $r = 2$ $a_n = a_1 \cdot r^{n-1}$

1, 4, 9, 16, ... not geom

3, 6, 12, 24, 48, ... geom $r = 2$ ✓

Example 1: Suppose $a_3 = 4$ and $a_7 = \frac{1}{4}$.

$16, \pm 8, 4, \pm 2, 1, \pm\frac{1}{2}, \frac{1}{4}$

$\frac{a_7}{a_3} = r^4 \qquad a_3 = 4$

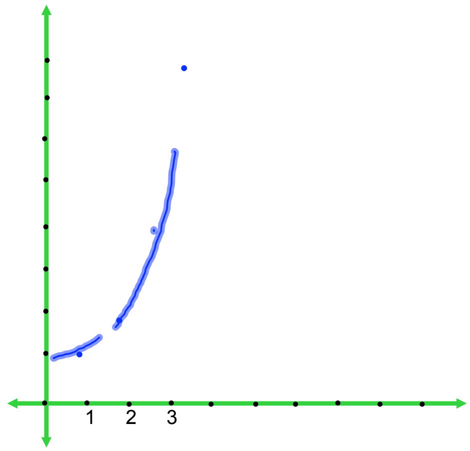
$a r^2 = 4$

$\frac{1}{16} = \frac{1/4}{4} = r^4 \qquad a \cdot \frac{1}{4} = 4$

$a = 16$

$r = \frac{1}{2}$ or $-\frac{1}{2}$

How would you describe the graph of a geometric sequence?



1, 2, 4, 8, 16, ...

$a = 1$

$r = 2$

Example 2:

Example 3:

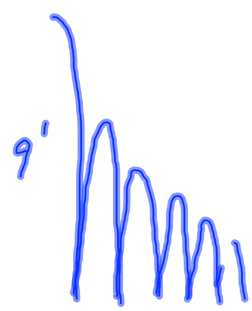
Suppose a ball is dropped from a height of 9 feet. The elasticity of the ball is such that it bounces up two-thirds of the distance that it has fallen. If this elasticity property remains in effect, how high will the all bounce after hitting the ground ten times?

6, 4, ...

$a_1 = 6$

$r = 2/3$

$a_{10} = a_1 \cdot r^9 = 6\left(\frac{2}{3}\right)^9 \approx 0.156$ ft



A *finite geometric series* is the sum S_n of the first n terms of a finite geometric sequence.

$$Sn = a_1 + (a_1 r) + (a_1 r^2) + (a_1 r^3) + \ldots + (a_1 r^{(n-1)})$$

S_n can be found by computing $S_n = \frac{a(1-r^n)}{1-r}$ ✓

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$
$$r \cdot S_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$$
$$S_n - rS_n = a - ar^n$$
$$S_n(1-r) = a(1-r^n) \Rightarrow S_n = \frac{a(1-r^n)}{1-r}$$

Example 4:

Find a formula for the n^{th} partial sum of the geometric series $3 + 6 + 12 + \ldots$ ✓
Use the formula to compute S_6.

$a = 3$
$r = 2$

$$S_n = \frac{a(1-r^n)}{1-r}$$
$$S_n = \frac{3(1-2^n)}{1-2} = 3(2^n - 1)$$
$$S_6 = 3(2^6-1) = 3 \cdot 63 = 189$$

Example 5:

a) Use the summation notation to write this series, determine a formula for the nth partial sum and find the sixth partial sum using the formula:

$$S_n = \frac{a(1-r^n)}{1-r}$$

$a = 1$
$r = 0.7$

a) $1 + 0.7 + 0.49 + 0.343 + \ldots$

$$S_n = \frac{1(1-0.7^n)}{1-0.7} = \frac{1-(0.7)^n}{0.3} = \frac{10}{3}\left(1-(0.7)^n\right)$$

$0.3 = 3/10$

$$S_6 = \frac{10}{3}\left(1-(0.7)^6\right) \approx 2.94$$

b) $\sum_{j=1}^{10} 2(0.1)^j = \frac{a(1-r^n)}{1-r} = \frac{.2(1-(0.1)^n)}{1-0.1} = \frac{.2(1-(0.1)^n)}{.9} = \frac{2(1-(0.1)^n)}{9}$

$a_1 = 2(0.1)^1 = .2$
$r = 0.1$
$n = 10$

$$S_{10} = \frac{2(1-(0.1)^{10})}{9} = \frac{2(0.999\,999\,999\,9)}{9}$$
$$= 0.222\,222\,222\,2$$

There is a mistake in this example. The exponent should be 10, not 9. This adds an extra 2 to the answer.

If the common ratio is between -1 and 1 ($|r| < 1$) in an infinite geometric series, the sum will converge to a finite sum. This is because r^n approaches zero as n increases without bound.

The formula for an infinite sum is:

$-1 < r < 1$

$$S_\infty = \sum_{j=1}^{\infty} a\, r^j = \frac{a}{1-r}$$

Where a is the first term, a_1 and $|r| < 1$

Example 6:

Compute the infinite sum of the two previous examples:

a) $1 + 0.7 + 0.49 + 0.343 + \ldots$

$a = 1$

$r = 0.7$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-0.7} = \frac{1}{0.3} = \frac{10}{3} = 3\tfrac{1}{3}$$

b) $\sum_{j=1}^{\infty} 2(0.1)^j = \frac{a}{1-r} = \frac{0.2}{1-0.1} = \frac{0.2}{0.9} = 2/9$

$r = 0.1$

$-1 < r < 1$

Example 7:

In the example of the bouncing ball dropped from a height of 9 feet and bouncing up two-thirds of the previous distance on each bounce, what is the total distance it has traveled after bouncing ten times?

$$9 + 2(6) + 2\cdot\left(\tfrac{2}{3}\cdot 6\right) + \cdots + 2\left(\left(\tfrac{2}{3}\right)^8\cdot 6\right)$$

$$= 9 + 2\sum_{i=1}^{9} 6\cdot\left(\tfrac{2}{3}\right)^{i-1} = 9 + 2\left(\frac{6\left(1-\left(\frac{2}{3}\right)^9\right)}{1-2/3}\right) = \approx 43.6\text{ ft}$$

$a = 6 \quad n = 9$

$r = 2/3$

If it really could bounce indefinitely, (an infinite number of times) how far would it have traveled?

Example 8:

In the last two lessons, you decided to save for your trip to Europe. You opened a savings account with $1.00 and on each subsequent day, you deposited a dollar more than on the previous day.

Now you get really brave and each day you deposit twice the amount you did on the previous day, starting with $1.00 on day 1. How much will you deposit on the 30th day? What is the total amount in the account on day 30?

$\$1 + \$2 + \$4 + \cdots$ 30 days

$a = 1, \quad r = 2, \quad n = 30$