CHAPTER 9: SEQUENCES AND SERIES

9.1 Sequences and Series

In section 9.1 you will learn to:

- Use sequence notation to write the terms of a sequence.
- Use factorial notation.
- Use summation notation to write sums.
- Find the sums of infinite series.
- Use sequences and series to model and solve real-life problems.

\[
\frac{1}{4!} \quad \sum_{k=2}^{8} \frac{1}{k^2} \quad \sum_{i=1}^{5} 2i
\]
What is a sequence?

Finite: 1, 2, 4, 8

\[ a_1, a_2, a_3, \ldots \]

Infinite: 1, 3, 5, 7, ..., 2n-1, ...

\[ a_n = 2n-1 \]

\( n=1 \rightarrow a_1 = 2(1)-1 = 1 \)

\( n=2 \rightarrow a_2 = 2(2)-1 = 3 \)

A sequence is a function with the domain a subset of the natural numbers.
Example 1:

a) Write the first four terms of this sequence: \( a_n = n^2 + 1 \)

\[
\begin{align*}
  a_1 &= 1^2 + 1 = 2 \\
  a_2 &= 2^2 + 1 = 5 \\
  a_3 &= 3^2 + 1 = 10 \\
  a_4 &= 17
\end{align*}
\]

b) Write the first four terms of this sequence: \( b_n = (-1)^{n+1}(10n + 3) \)

\[
\begin{align*}
  b_1 &= (-1)^2(10+3) = 13 \\
  b_2 &= (-1)^3(10\cdot2+3) = -23 \\
  b_3 &= 33 \\
  b_4 &= -43
\end{align*}
\]
Example 2: Find a formula for the \( n \)th term in each of these sequences, then use the formula to find the 10th term.

a) 2, 4, 6, 8, 10, ...

\[
a_n = 2n
\]

\[
a_{10} = 2(10) = 20
\]

b) 3, -6, 12, -24, ...

\[
b_n = (-1)^n \cdot (3 \cdot 2^{n-1})
\]

\[
b_{10} = (-1)^{10} \cdot (3 \cdot 2^9) = (1) \cdot (3 \cdot 2^9)
\]
Some sequences are defined **recursively**. One or more initial terms are given and subsequent terms are defined using the previous terms.

Example 3:

\[ a_1 = 2 \]
\[ a_n = 3a_{n-1} + 1 \text{ for each } n > 1 \]

What are the first four terms?

\[ a_1, a_2, a_3, a_4, \ldots \]
\[ 2, 7, 22, 67 \]

\[ a_1 = 2 \]
\[ a_2 = 3(2) + 1 = 7 \]
\[ a_3 = 3(7) + 1 = 22 \]
\[ a_4 = 3(22) + 1 = 67 \]
Example 4:

The **Fibonacci Sequence**

\[
\begin{align*}
a_1 &= 1 & \checkmark \\
a_2 &= 1 & \checkmark \\
a_n &= a_{n-1} + a_{n-2} & \checkmark \\
\end{align*}
\]

\[a_3 = a_1 + a_2 = 1 + 1 = 2\]

List five terms:

\[a_1, a_2, a_3, a_4, a_5\]

\[1, 1, 2, 3, 5, 8\]
Factorials are often used in sequence definitions.

We define \( n \) factorial (written \( n! \)) to be:

\[
    n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n
\]

\[
    4! = \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} = 24
\]

0! is defined to be 0! = 1

Example 5:

Evaluate these expressions:

a) \[
    \frac{8!}{2! \cdot 6!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{3 \cdot 2}{2} = 3
\]

b) \[
    \frac{(n+1)!}{(n-1)!} = \frac{(n+1) \cdot (n) \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1}{(n-1) \cdot \ldots \cdot 2 \cdot 1} = \frac{n \cdot (n+1)}{1} = n^2 + n
\]

It is often convenient to recognize the factorials of the first five or six natural numbers.

\[
    1! = 1 \quad 5! = 120
\]

\[
    2! = 2 \quad 6! = 720
\]

\[
    3! = 6
\]

\[
    4! = 24
\]
Example 6:

Write the first four terms of these sequences:

a) \( a_n = \frac{1}{n!} \)

\[
\begin{align*}
  a_1 &= \frac{1}{1!} = 1 \\
  a_2 &= \frac{1}{2!} = \frac{1}{2} \\
  a_3 &= \frac{1}{3!} = \frac{1}{6} \\
  a_4 &= \frac{1}{4!} = \frac{1}{24} \\
\end{align*}
\]

b) \( b_n = \frac{n}{(n+2)!} \)

\[
\begin{align*}
  b_1 &= \frac{1}{3!} = \frac{1}{6} \\
  b_2 &= \frac{2}{4!} = \frac{2}{24} \\
  b_3 &= \frac{3}{5!} = \frac{3}{120} \\
  b_4 &= \frac{4}{6!} = \frac{4}{720} \\
\end{align*}
\]
A **series** is the sum of the terms in a sequence. The sum of the first \( n \) terms of a sequence is the \( n^{th} \) **partial sum** \( S_n \).

The 5\(^{th} \) partial sum of the sequence of odd numbers is \( S_5 = 1 + 3 + 5 + 7 + 9 = 25 \).

For an arbitrary sequence \( a_1, a_2, a_3, \ldots, a_{100} \), the corresponding series is

\[ a_1 + a_2 + a_3 + \ldots + a_{100}. \]

We abbreviate this sum using the Greek letter \( \Sigma \) (sigma):

\[ \sum_{i=1}^{100} a_i = a_1 + a_2 + a_3 + \ldots + a_{100}. \]

The subscript \( i=1 \) and superscript 100 written above and below sign indicate which terms begin and end the series. The index \( i \) is not unique, but is sometimes replaced using \( j, k, \) etc.

Express \( 3^2 + 4^2 + 5^2 + 6^2 \) using the sigma.

\[ S = \sum_{j=3}^{6} j^2 \]
Example 7:

Find the sum of these series by adding the terms:

a) \( \sum_{j=1}^{5} (1+3j) \)  
\( a_j = 1 + 3j \)  
\( j = 1 \rightarrow j = 5 \)  
\( 4 + 7 + 10 + 13 + 16 = 50 \)

b) \( \sum_{k=0}^{2} \frac{(-1)^k}{2k} \)  
\( \frac{1}{1} \rightarrow k \)  
\( \frac{1}{2} \rightarrow k \)  
\( \frac{1}{4} \rightarrow k \)  
\( \frac{1}{8} \rightarrow k \)  
\( \frac{1}{16} \rightarrow k \)  
\( \frac{1}{32} \rightarrow k \)  
\( 1 - \frac{1}{2} - \frac{1}{4} = \frac{3}{4} \)

Use summation notation to abbreviate this series:

\[ \sum_{i=1}^{99} \frac{1}{i(i+1)} \]
Example 8:

You’re a clever student. You’ve decided to save your money for a trip to Europe, but it will be expensive. You’ve decided to open a savings account today with $1. You plan to add more each day, 7 days a week, by depositing one more dollar each day than you did the previous day. Use summation notation to express the total amount you will have contributed at the end of one year:

\[
1 + 2 + 3 + 4 + \cdots + 365 = \\
\sum_{j=1}^{365} j
\]