CHAPTER 8: MATRICES AND DETERMINANTS

In section 8.1 you will learn to:

- Write a matrix and identify the order.
- Perform elementary row operations on matrices
- Use matrices and Gaussian elimination (row-echelon form) to solve systems of linear equations.
- Use matrices and Gauss-Jordan elimination (reduced row-echelon form) to solve systems of linear equations.

Definition of a matrix:

matrix

$$[a_{ij}] = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} matrix

n = total # of cols

m = total # of rows

entry → a_{ij}; one term in matrix; i = the row and j = the col

order → size; $m \times n$ = # rows × # cols

row matrix → $[a_{11}\ a_{12}\ a_{13} \cdots a_{1n}]$ matrix w/ only one row $(1 \times n)$

column matrix → $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}$ $(m \times 1)$

We will use matrices to solve linear systems of equations.

<u>system</u>

$$\begin{cases} 3x - 2y + z = 5 \\ x + y + 2z = 1 \\ -x \quad 0 \quad - z = 0 \end{cases}$$

3 variables, 3 eqns.

$$\xrightarrow{1}$$

<u>augmented matrix</u>

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 5 \\ 1 & 1 & 2 & 1 \\ -1 & 0 & -1 & 0 \end{array}\right] \quad (3 \times 4)$$

2

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

coefficient matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

variable matrix

$$C = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

constant matrix

AX = C

Example 1 -- What is the size (order) of these matrices?
Are any of them square?

a) $\begin{bmatrix} -2 & 5 & 1 \\ 7 & 6 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 2 & 3 & 1 & 5 \\ 4 & 4 & 7 & 4 & 4 & -1 \\ 9 & 8 & 7 & 6 & 5 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

(a) 2×3
not-square
(rectangular)

(b) 4×6
rectangular

(c) 2×2
square

Gaussian Elimination:

Row-echelon form →

- All zero rows at the bottom
- Has a leading 1 in every nonzero row
- All entries below the leading 1 are zero.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 3 & 1 & 5 \\ 0 & 0 & 1 & 3 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Gauss-Jordan Elimination

Reduced row-echelon form →

- row-echelon form

and

- all entries above leading 1 are zero

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 4 \end{array}\right]$$

Example 2 - Indicate if these matrices are in

- ⓐ - row-echelon form
- ⓑ - reduced row-echelon form
- ⓒ - neither

a) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 5 \\ 1 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 5 & 6 & 3 \\ 0 & 1 & 2 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 7 & 9 \\ 0 & 1 & 3 & 5 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

(a) 3x3 neither

(b) 2x4 row-echelon form

(c) 3x4 neither

(d) 3x4 reduced row-echelon form

Example 3: a) Write the system of equations represented by this augmented matrix.
b) Write this matrix in row-echelon form.
c) Back-substitute to solve.

$$\left[\begin{array}{ccc:c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right]$$

(a) $\begin{cases} x \quad -3z = -2 \\ 3x + y - 2z = 5 \\ 2x + 2y + z = 4 \end{cases}$

(b) Elementary Row Operations:

1. Exchange any two rows.
2. Multiply any row by nonzero constant.
3. Add a nonzero multiple of one row to another row, replacing one of the rows w/ the result. (elimination step)

$$(-3)\left[\begin{array}{ccc:c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right]$$

get those to be zeros

(temp)

$$\left[\begin{array}{ccc:c} -3 & 0 & 9 & 6 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right]$$

$$\begin{array}{cccc} -2 & 0 & 6 & 4 \end{array}$$

$$(-2)\left[\begin{array}{ccc:c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 2 & 2 & 1 & 4 \end{array}\right]$$

$$\begin{array}{cccc} 0 & -2 & -14 & -22 \end{array}$$

$$(-2)\left[\begin{array}{ccc:c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right]$$

$$\left(\frac{1}{-7}\right)\left[\begin{array}{ccc:c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x - 3z = -2$

$y + 7z = 11$

$z = 2$

$y + 7(2) = 11$

$y + 14 = 11$

$y = -3$

$x - 3(2) = -2$

$x - 6 = -2$

$x = 4$

$\boxed{(4, -3, 2)}$