Partial Fraction Decomposition

In this section you will learn to:

- Recognize partial fraction decompositions of rational expressions.
- Find partial fraction decompositions of rational expressions.

Let's add two rational expressions together.

$$\frac{3}{x-2}+\frac{5}{x+3} = \frac{3}{(x-2)}\left(\frac{x+3}{x+3}\right)+\frac{5}{(x+3)}\left(\frac{x-2}{x-2}\right)$$

LCD $=(x-2)(x+3)$

$$= \frac{3(x+3)+5(x-2)}{(x+3)(x-2)} = \frac{3x+9+5x-10}{(x+3)(x-2)}$$

$$= \boxed{\frac{8x-1}{(x+3)(x-2)}}$$

Now, let's undo what we just did. Start with the answer and determine the question.

(PFD)

start: $\dfrac{8x-1}{(x+3)(x-2)} = \dfrac{A}{x+3} + \dfrac{B}{x-2}$

if denominator is linear factor, numerator is a constant

multiply both sides by denominator on the left:

$$\frac{\cancel{(x+3)}\cancel{(x-2)}(8x-1)}{\cancel{(x+3)}\cancel{(x-2)}} = \frac{A\cancel{(x+3)}(x-2)}{\cancel{(x+3)}} + \frac{B(x+3)\cancel{(x-2)}}{\cancel{(x-2)}}$$

simplify: $8x-1 = A(x-2)+B(x+3)$

this eqn must be true for all x-values so we can plug in specific x-values to solve for A & B.

$x=2$: $8(2)-1 = 0A + B(5)$

$15 = 5B \Rightarrow \boxed{B=3}$

$x=-3$: $8(-3)-1 = A(-5)+0B$

$-25 = -5A \Rightarrow \boxed{A=5}$

$$\boxed{\frac{8x-1}{(x-2)(x+3)} = \frac{5}{x+3} + \frac{3}{x-2}}$$

To Decompose $\frac{N(x)}{D(x)}$ into Partial Fractions:

$N(x)$, $D(x)$ both polynomials

- Divide when improper. (when degree of $N(x) \geq$ degree of $D(x)$)
- Factor the denominator. (completely)
- Set up appropriate terms as outlined in the following examples.

$$\frac{x^3+2x^2-x+1}{x^2+3x-4}$$

(this is improper)

①
$$\begin{array}{r} x-1 \\ x^2+3x-4 \,\big)\, \overline{x^3+2x^2-x+1} \\ -(x^3+3x^2-4x) \\ \hline -x^2+3x+1 \\ -(-x^2-3x+4) \\ \hline 6x-3 \end{array}$$

$$\Rightarrow \frac{x^3+2x^2-x+1}{x^2+3x-4} = x-1+\frac{6x-3}{x^2+3x-4}$$

② do PFD on remainder rational expression (proper)

$$\frac{6x-3}{x^2+3x-4} = \underbrace{\frac{6x-3}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}}$$

multiply both sides by denominator on left:

$$6x-3 = A(x-1)+B(x+4)$$ true for all x values.

$x=1$: $6(1)-3 = 0A + B(5)$

$3 = 5B \Rightarrow B = 3/5$

$x=-4$: $6(-4)-3 = A(-5)+0$

$-27 = -5A$

$A = 27/5$

$$\Rightarrow \boxed{\frac{x^3+2x^2-x+1}{x^2+3x-4} = x-1+\frac{27/5}{x+4}+\frac{3/5}{x-1} = x-1+\frac{27}{5(x+4)}+\frac{3}{5(x-1)}}$$

Distinct Linear Factors

EX 1: Write the partial fraction decomposition of $\dfrac{x+2}{x^3-9x}$.

① it is a proper rational expression
(don't need long division)

② $$\frac{x+2}{x(x^2-9)} = \underbrace{\frac{x+2}{x(x-3)(x+3)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+3}}$$

$$x+2 = A(x-3)(x+3) + Bx(x+3) + C(x)(x-3)$$

$x=0$: $0+2 = A(-3)(3) + 0B + 0C$

$2 = -9A \Rightarrow A = -2/9$

$x=3$: $3+2 = 0A + B(3)(6) + 0C$

$5 = 18B \Rightarrow B = 5/18$

$x=-3$: $-3+2 = 0A + 0B + C(-3)(-6)$

$-1 = -18C \Rightarrow C = 1/18$

$$\Rightarrow \frac{x+2}{x(x-3)(x+3)} = \frac{-2/9}{x} + \frac{5/18}{x-3} + \frac{1/18}{x+3}$$

$$= \boxed{\frac{-2}{9x} + \frac{5}{18(x-3)} + \frac{1}{18(x+3)}}$$

Repeated Linear Factors

EX 2: Write the partial fraction decomposition of $\frac{2x-3}{(x-1)^2}$.

① this is proper rational expression.
(no long division)

② $$\frac{2x-3}{(x-1)^2} = \underbrace{\frac{2x-3}{(x-1)(x-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2}}$$

$$2x-3 = A(x-1) + B$$

$x=1$: $2(1)-3 = A(0) + B$

$-1 = B$

$x=0$: $2(0)-3 = A(-1) + B$

$-3 = -A + -1$

$-2 = -A$

$A = 2$

$$\Rightarrow \boxed{\frac{2x-3}{(x-1)^2} = \frac{2}{x-1} + \frac{-1}{(x-1)^2}}$$

Distinct Linear and Quadratic Factors

EX 3: Write the partial fraction decomposition of $\frac{3x^2+4x+4}{x^3+4x}$.

(proper)

$$\frac{3x^2+4x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

"prime quadratic factor"

$$3x^2+4x+4 = A(x^2+4)+(Bx+C)x$$

different technique to find A and B:
"equating like coefficients"

$$3x^2+4x+4 = Ax^2+4A+Bx^2+Cx$$

$$3x^2+4x+4 = x^2(A+B)+x(C)+(4A)$$

since this eqn must be true for all x values then the # of x^2 on Left = # of x^2 on right, etc.

$\boxed{x^2}$ $3=A+B$

$\boxed{x}$ $4=C$

$\boxed{\text{const}}$ $4=4A$

$A=1$

$3=1+B$

$B=2$

$$\boxed{\frac{3x^2+4x+4}{x^3+4x} = \frac{1}{x} + \frac{2x+4}{x^2+4}}$$

Repeated Quadratic Factors

EX 4: Write the partial fraction decomposition of $\dfrac{x^3-4x^2+2x-6}{x\,(x^2+2)^2}$.

(proper rational expression)

$$\frac{x^3-4x^2+2x-6}{x(x^2+2)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+2}+\frac{Dx+E}{(x^2+2)^2}$$

$$x^3-4x^2+2x-6=A(x^2+2)^2+(Bx+C)(x)(x^2+2)+(Dx+E)x$$

(use equating like coefficients technique)

combo

$x=0$: $-6=A(2^2)+0+0$

$\Leftrightarrow 4A=-6 \Leftrightarrow A=-3/2$

$$x^3-4x^2+2x-6=\tfrac{-3}{2}(x^2+2)^2+(Bx^2+Cx)(x^2+2)+Dx^2+Ex$$

$$x^3-4x^2+2x-6=-\tfrac{3}{2}(x^4+4x^2+4)+Bx^4+2Bx^2+Cx^3+2Cx+Dx^2+Ex$$

$$0x^4+x^3-4x^2+2x-6=x^4(B-3/2)+x^3(C)+x^2(-\tfrac{3}{2}(4)+2B+D)+x(2C+E)+(-\tfrac{3}{2}(4))$$

$\boxed{x^4}$: $0=B-3/2$, $\boxed{B=3/2}$

$\boxed{x^3}$: $\boxed{1=C}$

$\boxed{x^2}$: $-4=\tfrac{-3}{2}(4)+2B+D$

$-4=-6+2(3/2)+D$

$-4=-6+3+D$

$\boxed{D=-1}$

$\boxed{x}$: $2=2C+E$

$2=2+E$

$\boxed{E=0}$

$\boxed{\text{const}}$: $-6=-6$ ✓

$$\frac{x^3-4x^2+2x-6}{x(x^2+2)^2}=\frac{-3/2}{x}+\left(\frac{2}{2}\right)\left(\frac{3/2\,x+1}{x^2+2}\right)+\frac{-1x+0}{(x^2+2)^2}$$

$$=\boxed{\frac{-3}{2x}+\frac{3x+2}{2(x^2+2)}-\frac{x}{(x^2+2)^2}}$$