3.3 Properties of Logarithms

Properties of Logarithms

In section 3.3 you will learn to:

• Use properties to evaluate or rewrite logarithmic expressions.
• Use properties of logarithms to expand or condense logarithmic expressions.
• Use the change of base formula to rewrite and evaluate logarithmic expressions.
• Use logarithmic functions to model and solve real-life problems.
Properties of Logarithms

Your calculator has only two keys that compute logarithmic values.

\[ \log x \text{ means } \log_{10} x \]

\[ \ln x \text{ means } \log_e x \]

Suppose you need to compute a logarithm in some other base, \( a \)

\[ \log_a x = y \]

\[ a^y = x \]

\[ \log (a^y) = \log x \]

\[ y \log a = \log x \]

\[ y = \frac{\log x}{\log a} = -\log_a x \]

Change of base formula:

\[ \log_a x = \frac{\log_b x}{\log_b a} \]

Examples:

a) \( \log_2 254 = \frac{\log 254}{\log 2} \approx 7.989 \)

b) \( \log_6 0.008 = \frac{\ln 0.008}{\ln 6} \approx -2.695 \)

\[ \log_6 0.008 = \frac{\log 0.008}{\log 6} \approx -2.695 \]
3.3 Properties of Logarithms

Since a logarithm is an exponent, the properties of logarithms are just like the properties of exponents.

<table>
<thead>
<tr>
<th>Exponents</th>
<th>Logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^0 = 1$</td>
<td>$\log_a 1 = 0$</td>
</tr>
<tr>
<td>$a^1 = a$</td>
<td>$\log_a a = 1$</td>
</tr>
<tr>
<td><strong>Product:</strong></td>
<td>[ \log_a (uv) = \log_a u + \log_a v ]</td>
</tr>
<tr>
<td>[ a^m \cdot a^n = a^{m+n} ]</td>
<td>[ \log_a \frac{u}{v} = \log_a u - \log_a v ]</td>
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<tr>
<td><strong>Quotient:</strong></td>
<td>[ \log_a u^n = n \log_a u ]</td>
</tr>
<tr>
<td>[ \frac{a^m}{a^n} = a^{m-n} ]</td>
<td></td>
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<tr>
<td><strong>Power:</strong></td>
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<tr>
<td>[ (a^m)^n = a^{mn} ]</td>
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</tr>
<tr>
<td><strong>Inverse properties:</strong></td>
<td>[ \log_a (a^x) = x ]</td>
</tr>
<tr>
<td><strong>One-to-one properties:</strong></td>
<td>[ \log_a x = \log_a y \implies x = y ]</td>
</tr>
</tbody>
</table>
Let's apply the properties of logarithms.

a) \( \log_4 5 + \log_4 6 = \log_4 30 \)

b) \( \log (12a) - \log (2a) = \log \left( \frac{12a}{2a} \right) = \log 6 \)

c) \( \log_4 x^4 = 4 \log_4 x = \log_4 x \)

d) \( e^{\ln(5x)} = 5x \)

e) \( \log_{10} (x^2) = \frac{\log_{10} x^2}{\log_{10} 10} = x^2 \)
In solving equations, it will be helpful to expand and condense logarithmic expressions.

Expand these:

a) \( \log_4 5x^3y = \log_4 5 + \log_4 (x^3) + \log_4 y = \)

\[ = \log_4 5 + 3 \log_4 x + \log_4 y \]

b) \( \ln \frac{\sqrt{3x-5}}{7} = \ln \sqrt{3x-5} - \ln 7 = \)

\[ = \frac{1}{2} \ln (3x-5) - \ln 7 \]

c) \( \log \left( \frac{b^3}{1+a^2} \right)^5 = 5 \log \left( \frac{b^3}{1+a^2} \right) = \)

\[ = 5 \left[ 3 \log b - \log (1+a^2) \right] \]
Condense these into a single logarithmic expression:

a) \( \frac{1}{2} \log x + 3 \log (x+1) = \log \left( \sqrt{x} (x+1)^3 \right) = \log \left( \frac{\sqrt{x}}{x+1} \right) \)

b) \( 2 \ln (x+2) - \ln x = \ln \left( \frac{(x+2)^2}{x} \right) \)
Suppose we know that \( \log_b 2 = 0.41 \) and \( \log_b 3 = 0.54 \), use the properties of logarithms to find:

a) \( \log_b 6 = \log_b (2 \cdot 3) = \log_b 2 + \log_b 3 = 0.41 + 0.54 = 0.95 \)

b) \( \log_b 2/9 = \log_b \sqrt[3]{3} = \frac{1}{3} \log_b 3 = \frac{1}{3} \cdot 0.54 = 0.18 \)

\( \log_b 2 - \log_b 9 = \log_b 2 - 2 \log_b 3 = 0.41 - 0.94 = -0.53 \)
Logarithms are useful in reporting a broad range of data by converting it into a more manageable form. Consider the intensity of earthquakes.

Let \( I_0 = \) the intensity of a "standard" earthquake that is agreed upon as minimal (barely detectable.)

Let \( I = \) The intensity of a much larger earthquake.

The magnitude \( M \) of the latter quake \( I \) relative to \( I_0 \) is defined by

\[
M = \log \frac{I}{I_0}
\]

You may have heard of the Richter scale that measures the intensity of an earthquake.

What is the magnitude \( M \) of an earthquake measured to be 10,000 times more intense than a standard quake?

\[
M = \log \frac{10,000 \times I_0}{I_0} = \log 10,000 = 4.0
\]
Example:

On October 17, 1989 a major earthquake struck the San Francisco Bay area only minutes before Game 3 of the World Series in Candlestick Park. Its intensity was measured as 7.1 on the Richter scale.

How many times more intense was it than a minimal quake?

a) 12,500 times more intense?

b) 1,250,000 time more intense?

c) 12,500,000 times more intense?

\[
M = \log \frac{I}{I_0}
\]

\[
7.1 = \log \frac{I}{I_0}
\]

\[
10^{7.1} = \frac{I}{I_0}
\]

\[
I = 10^{7.1} (I_0) = 12,500,000 I_0
\]