2.5 Finding the zeros of polynomial functions

We will learn how to:

• Determine the number of zeros of polynomial functions
• Find rational zeros of polynomial functions
• Find conjugate pairs of complex zeros
• Find zeros of polynomials by factoring
• Write a polynomial function given the roots.
\[ P(x) = a_1 x^n + a_2 x^{n-1} + \ldots + a_{n-1} x + a_n \]

Factored form \( a(x - r_1)(x - r_2) \ldots (x - r_n) \)

The roots are \( r_1, r_2, \ldots, r_n \)
Rational Root Theorem:

\[ P(x) = (2x-5)(3x+2)(x-1) \]

Has roots: \( x = \frac{5}{2}, -\frac{2}{3}, 1 \)

\[ P(x) = 6x^3 - 17x^2 + x + 10 \]

If \( P(x) \) has any rational roots, they will be of the form \( \frac{p}{q} \) where \( p \) and \( q \) have no common factors other than 1 and where \( p \) is a factor of the constant term and \( q \) is a factor of the leading coefficient.
This allows us to attempt to break higher degree polynomials down into their factored form and determine the roots of a polynomial.

Example 1: Factor completely and determine the roots of this polynomial.

\[ P(x) = x^3 + 3x^2 + x - 2 \]

1) set of \( p_s \) \( \pm 1, \pm 2 \)
2) set of \( q_s \) \( \pm 1 \)
3) possible roots of \( P(x) \) \( \pm 1, \pm 2 \)

4) Test each possible root using synthetic division:

\[
\begin{array}{c|cccc}
1 & 1 & 3 & 1 & -2 \\
\hline
 & 1 & 4 & 5 & 3
\end{array}
\]

\[
\frac{1}{1} \frac{4}{1} \frac{5}{1} \frac{3}{not}
\]

\[
\begin{array}{c|cccc}
2 & 1 & 3 & 1 & -2 \\
\hline
 & 2 & 10 & 22 & 15
\end{array}
\]

\[
\frac{2}{1} \frac{2}{1} \frac{11}{not}
\]

\[
\begin{array}{c|cccc}
-1 & 1 & 3 & 1 & -2 \\
\hline
 & -1 & 2 & 3 & 0
\end{array}
\]

\[
\frac{-1}{-1} \frac{2}{-1} \frac{3}{0}
\]

\[
\frac{1}{x^3 - x - 1} \rightleftharpoons (x+2)
\]

\[
X^3 + 3x^2 + x - 2 = (x+2)(x^2 + x - 1)
\]

\[
x = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}
\]

Roots: -2, \( \frac{-1 + \sqrt{5}}{2} \), \( \frac{-1 - \sqrt{5}}{2} \)
Example 2: Find the roots and write in factored form:

\[ y = 9x^4 - 3x^3 + x^2 - 8x + 4 \]

Roots: \( \frac{2}{3} \) (double)

\[
x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}}{2}
\]

\[ x = \frac{-1}{2} + \frac{\sqrt{3}}{2} \]
\[ x = \frac{-1}{2} - \frac{\sqrt{3}}{2} \]
\[-x = \frac{2}{3} \text{ (double root)} \]
Example 3:

Determine the roots and write in factored form:

\[ y = x^3 - 7x - 6 \]

Possible roots: \( \{1, 2, 3, 6\} \)

By inspection:

\[ 3 \mid 10 - 7 - 6 \]
\[ 3 \mid 9 \quad 6 \]
\[ 3 \mid 2 \, \checkmark \]

\[
(x-3)(x^2+3x+2) = (x-3)(x+1)(x+2)
\]

Factorized form:

\[
\chi = \frac{-3 \pm \sqrt{9 - 8}}{2} = \frac{-3 \pm 1}{2}
\]

- \( \frac{-3 + 1}{2} = -1 \, \checkmark \)
- \( \frac{-3 - 1}{2} = -2 \, \checkmark \)
- \( \chi = 3 \, \checkmark \)
Notice: as soon as you can get the factored form down to a quadratic, use the quadratic formula to find the other two roots. They may be complex. Complex roots will come in conjugate pairs. If $a + bi$ is a root, then $a - bi$ will be a root if the polynomial has integer coefficients.

Example 4: Factor and determine the roots:

$$y = x^3 + 4x^2 + 14x + 20$$

Factored form: $(x+2)(x^2+2x+10)$

Roots:
- $-2$
- $-1 + 6i$
- $-1 - 6i$

$$x^2 + 2x + 10 = 0$$
$$x = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 6i$$

1 real -
2 complex
Example 5:

Write a polynomial function with real coefficients of degree 4 which has these roots:

\[ 2i, -3, 1 \]

\[ (x-2i)(x+2i) = x^2 - 4i^2 \]
\[ (x^2+4)(x+3)(x-1) \]
\[ (x^3+9)(x+3) = x^3 + 3x^2 + 4x + 12 \]
\[ \cdot (x-1) \]

\[ x^4 + 3x^3 + 4x^2 + 12x \]
\[ -x^3 - 3x^2 - 4x - 12 \]

\[ x^4 + 2x^3 + x^2 + 8x - 12 \]