Section 2.4 Complex Numbers

- Use the imaginary unit, *i* to write complex numbers
- Add, subtract, multiply and divide complex numbers in standard form (a+b*i*)
- Use complex conjugates to write the quotient of two complex numbers in standard form
- Find complex solutions to polynomial equations

$3-2i$

The solution to this equation is simple: $x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

The solution to this equation is not so simple:
$x^2 + 9 = 0$

$x^2 = -9$

$x^2 = -9$

$x = \pm\sqrt{-9} = \pm\sqrt{9}\,i$

$\pm 3i$

A new term defined: $\sqrt{-1}$ is defined to be i.

So $\sqrt{-4} = \sqrt{-1}\cdot\sqrt{4} = 2i$

$\sqrt{-8} = \sqrt{4}\sqrt{2}\sqrt{-1} = 2\sqrt{2}\,i$

$\sqrt{-7} = \sqrt{7}\,i$

Let's look at some powers of i :

$i^2 = -1$

$i^3 = -1 \cdot i = -i$

$i^4 = (-i)i = -i^2 = -(-1) = 1$

$i^5 = i$

$i^6 = -1$

$i^7 = -i$

We see a pattern developing, so can find any power of i .

$i^{38} = (i)^{36} \cdot i^2$
$(i^4)^9 \cdot i^2$
$1^9 \cdot i^2 = i^2$

$i^{65} = i^{64} \cdot i$
$(i^4)^{16} \cdot i$
$\underline{\underline{i}}$

$i^{321} = i^{320} \cdot i^1$
$(i^4)^{80} \cdot i$
$\underline{\underline{i}}$

A complex number $a + bi$ has a real part a

and an imaginary part, bi.

Operations on these are not surprising.

To add two complex numbers, add the real parts and add the imaginary parts.
Subtraction is the same idea:

$$(3-2i) + (5-4i) = 8-6i$$

$$(3-2i) - (5-4i) = 3-2i-5+4i$$
$$\underline{-2+2i}$$

To multiply two complex numbers, you do what you would when multiply two binomials:

$$(3-2i) \cdot (5-4i) = \underbrace{15+8i^2}_{8(-1)} - 12i - 10i$$
$$\underline{7-22i}$$

Division of two complex numbers is a bit trickier.
We need to introduce another term.

The conjugate of a+ bi is a-bi.

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Write a conjugate for each of these:

-3 + 4i	2 - 5i	2 + i
conj. $-3-4i$	$2+5i$	$2-i$

Now multiply each of the conjugate pairs above, something amazing happens.

$(-3+4i)(-3-4i) = 9-16i^2 + 12i - 12i$

$\underline{25}$

$(2-5i)(2+5i) = 4-25i^2 + 10i - 10i$

$\underline{29}$

$(2+i)(2-i) = 4-i^2 - 2i + 2i$

$\underline{5}$

When you multiply a complex number by its conjugate, the result is a real number.

Now we can learn how to divide two complex numbers. To divide a+bi by c+di, multiply the numerator and denominator by the conjugate of the denominator.

$$(3-2i) \div (5-4i) = \frac{(3-2i)}{(5-4i)} \frac{(5+4i)}{(5+4i)} = \frac{15+8+12i-10i}{25+16}$$

$$\frac{23+2i}{41} = \frac{23}{41} + \frac{2}{41} i$$

Now we can write complex solutions to quadratic equations. Solve for the roots of each of these simplifying the radical expression as much as possible.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

a) $3x^2 - 2x + 5 = 0$

$$x = \frac{2 \pm \sqrt{4-4(3)(5)}}{6} = \frac{2 \pm \sqrt{-56}}{6} = \frac{2 \pm 2\sqrt{14}\,i}{6}$$

$$\underline{\frac{1}{3} \pm \frac{\sqrt{14}}{3} i}$$

b) $4x^2 + 6x + 3 = 0$

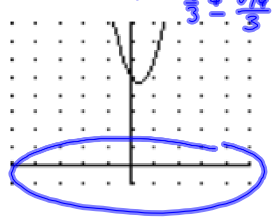
$$x = \frac{-6 \pm \sqrt{36-48}}{8} = \frac{-6 \pm \sqrt{-12}}{8} = \frac{-6 \pm 2\sqrt{3}\,i}{8}$$

$$-\frac{3}{4} \pm \frac{\sqrt{3}}{4} i$$

What does this mean about the x-intercepts of the graphs of these functions?

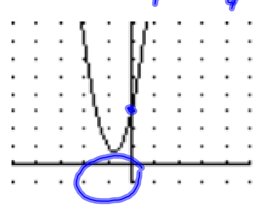
a) $y = 3x^2 - 2x + 5$

roots: $\frac{1}{3} \pm \frac{\sqrt{14}}{3} i$



b) $y = 4x^2 + 6x + \underline{3}$

roots $-\frac{3}{4} \pm \frac{\sqrt{3}}{4} i$



If the solutions to a quadratic equation are complex, then the quadratic does not have x-intercepts. It will not cross the x-axis.

Roots of a quadratic:

If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Are the roots of these real? $\sqrt{b^2-4ac} \rightarrow$ discriminant

If they are real, are they rational or irrational? Complex

$\sqrt{\ }$ i

a) $y = 3x^2 - x + 5$

$\sqrt{1-60}$

$\sqrt{-59} \rightarrow$ complex roots

b) $y = 3x^2 + 8x - 5$

$\sqrt{64+60}$

$\sqrt{124} = 2\sqrt{31}$

irrational roots

c) $y = x^2 - x + 2$

$\sqrt{1-8}$

$\sqrt{-7}$

not real

Complex roots

d) $y = 3x^2 - 8x - 3$

$\sqrt{64+36}$

$\sqrt{100} = 10$

rational roots