Composition of functions

Inverse functions
Today's objectives

• Define composition of functions
• Give examples of composing functions algebraically and by graphing
• Define inverse function
• Practice finding inverse function algebraically and by graphing
Beads and necklaces

• Few years ago I took up beading for fun. I would buy a bag of varied beads and found that I can make 14 necklaces from it.

\[ f(b) = 14b \]

• As my beading skills got better, I found that people liked my designs and are willing to pay for my necklaces. I started selling them at a local farmers' market for $9.50.

\[ g(n) = 9.5n \]

• I would like to know how much money I will make based on the number of bags of beads I buy.
Definition

- Let $f: A \to B$, $g: B \to C$ be two functions. Composition of $f$ and $g$ is a function, denoted by $g \circ f$, defined by:

\[ g \circ f(x) = g(f(x)) \]
Find \( g \circ f \) if

\[
f(x) = 7x - 2 \\
g(x) = x^2 - 2x
\]

\[
\begin{align*}
g \circ f(x) &= g(f(x)) = g(7x - 2) \\
&= (7x - 2)^2 - 2(7x - 2) \\
&= 49x^2 - 28x + 4 - 14x + 4 \\
&= 49x^2 - 42x + 8
\end{align*}
\]

\[
f(x) = x^2 - 2x \\
g(x) = 7x - 2
\]

\[
\begin{align*}
\circ f(x): g(f(x)) &= g(x^2 - 2x) \\
&= 7(x^2 - 2x) - 2 \\
&= 7x^2 - 14x - 2
\end{align*}
\]

\[g \circ f \neq f \circ g\]
The following functions can be written as \( g \circ f \). What are \( f \) and \( g \)?

\[
F(x) = \sqrt{x^2 - 2x + 1} \\
g(x) = \sqrt{x} \\
f(x) = x^2 - 2x + 1 \\
g \circ f(x) = g(f(x)) = g(x^2 - 2x + 1) = \sqrt{x^2 - 2x + 1} = F(x)
\]

\[
F(x) = \frac{x + 2}{x + 7} \\
g(x) = \frac{x}{x + 5} \\
f(x) = x + 2 \\
g \circ f(x) = g(f(x)) = g\left(\frac{x + 2}{x + 7}\right) = \frac{x + 2}{x + 7} = F(x)
\]
Graphing composition of functions

\( g \circ f(x) = g(f(x)) \)

\((x, g(f(x)))\) on the graph of \(g \circ f\)
If we did a whole bunch of points
Remember my beading problem?

- As my beading skills got better, I found that people liked my designs and are willing to pay for my necklaces. I started selling them at a local farmers' market for $9.50.

\[ g(n) = 9.5n \]

- I would like to know how many necklaces I need to make in order to earn $779.

\[
\begin{align*}
779 &= 9.5n \\
\frac{779}{9.5} &= n \\
82 &= n
\end{align*}
\]

What if I wanted to represent \( n \) in terms of my earnings?

\[ g(n) = 9.5n \]

\[ \frac{g(n)}{9.5} = n \]

\[ g(n) = m \]

\[ \frac{m}{9.5} = n \]

\[ h(m) = \# \text{ necklaces needed to make } $m \]

\[ h(m) = \frac{m}{9.5} \]

\[ h(g(m)) = m \]

\[ g(h(m)) = m \]
Interesting question

• If I have a function $f$ can I find function $g$ so that $g \circ f(x) = x$?
Inverse function

• If a function $f: A \rightarrow B$ has the property that each element of $B$ is the image of exactly one element of $A$ (we say $f$ is injective), then $f$ has an inverse function, $f^{-1}$

\[
\begin{align*}
    f \circ f^{-1}(x) &= x \\
    f^{-1} \circ f(x) &= x
\end{align*}
\]

• Horizontal line test: Function $f$ has an inverse if each horizontal line intersects the graph of $f$ in exactly one point.
Finding the inverse function

\[
\begin{array}{c|c}
 x & f(x) = y \\
1 & 4 \\
2 & 5 \\
3 & 6 \\
\end{array}
\quad
\begin{array}{c|c}
 x & f'(x) \\
4 & 1 \\
5 & 2 \\
6 & 3 \\
\end{array}
\]

\[y = f(x) \quad \rightarrow \quad \text{solve for } x \text{ so that you have } x \text{ in terms of } y\]
Finding expression for inverse

\[ f(x) = 2x + 1 \]
\[ y = 2x + 1 \]
\[ y - 1 = 2x \] \[ \therefore \] \[ f^{-1}(x) = \frac{x - 1}{2} \]
\[ \frac{y - 1}{2} = x \]

\[ g(x) = \frac{x - 3}{2x + 1} \]
\[ \frac{x - 3}{2x + 1} = y \] \[ \therefore (2x + 1) \]
\[ x - 3 = y(2x + 1) \]
\[ x - 3 = 2xy + y \]
\[ x - 2xy = y + 3 \]
\[ x(1 - 2y) = y + 3 \] \[ \therefore \] \[ (1 - 2y) \]
\[ x = \frac{y + 3}{1 - 2y} \]
\[ g^{-1}(x) = \frac{x + 3}{1 - 2x} \]

\[ h(x) = 2x^2 + 1 \]
\[ y = 2x^2 + 1 \] \[ \therefore \]
\[ y - 1 = 2x^2 \] \[ \therefore \]
\[ \frac{y - 1}{2} = x \] \[ \sqrt{\frac{y - 1}{2}} = x \]

h does not have an inverse
Finding the graph of inverse function

\[(a, f(a))\]

\[(f(a), a)\]

on the graph of \( f^{-1} \)