Name:______________________________

Student ID:______________________________

State Math Contest (Senior)

Instructions:

- Do not turn this page until your proctor tells you.
- Enter your name, grade, and school information following the instructions given by your proctor.
- Calculators are **not** allowed on this exam.
- This is a multiple choice test with 40 questions. Each question is followed by answers marked a), b), c), d), and e). Only one answer is correct.
- Mark your answer to each problem on the bubble sheet Answer Form with a #2 pencil. Erase errors and stray marks. Only answers properly marked on the bubble sheet will be graded.
- **Scoring:** You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
- You will have 2 hours and 30 minutes to finish the test.

1. Which of the following could not be the lengths of the sides of a triangle?

a) 5, 7, 8 b) 2, 4, 5 c) 4, 7, 10

d) 1, 1, 1 e) 3, 6, 10

Solution:

Correct answer: e
Three numbers represent the lengths of sides of a triangle if and only if they satisfy the triangle inequality. Notice that $3+6<10$ implies doesn't satisfy the triangle inequality.

2. A bumble bee is traveling back and forth between the front end of two trains moving towards each other. If the trains start 90 miles away from each other and one train is going 10 miles per hour while the other train is going 20 miles an hour and the bumble bee is traveling 100 miles per hour, how many miles does the bumble bee travel before being smashed by the two trains colliding?

a) 450 miles b) 90 miles c) 300 miles

d) 180 miles e) 270 miles

Solution:

Correct answer: c
The trains start 90 miles apart and collectively are going 30 mile per hour. It will take 3 hours for the trains to collide. Since the bubble bee is traveling 100 miles per hour and travels for 3 hours, the bee travels 300 miles.

3. Four people's car keys are accidentally dropped into a pond. When the keys are fished out, they look alike except for the notches, so the four keys are returned in *random* order to the four owners. What is the probability that none of the owners will be able to get into his or her own car later?

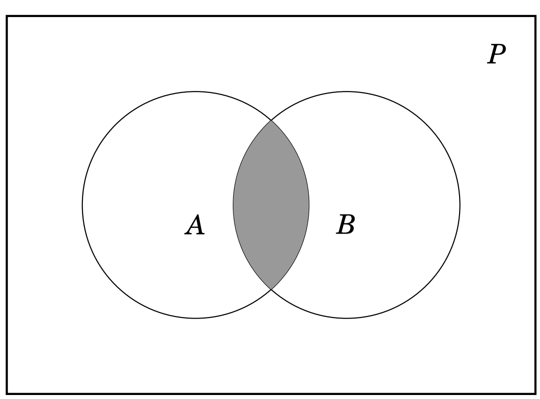
a) 1/4 b) 3/8 c) 5/12

d) 1/2 e) 5/8

Solution:

Correct answer: b
Keys 1234 have 24 permutations.
Possible totally wrong orders: 2143 2341 2413 3142 3412 3422 4123 4312 4321

4. Which of the following expressions indicate the shaded region. ($\overline{A}$ is the complement of A relative to P.)



I. $A \cup B$ II. $A \cap B$ III. $P \cup (A \cup B)$ IV. $P \cap (A \cup B)$

V. $P \cap (A \cap B)$ VI. $P \cup (A \cap B)$ VII. $\overline{\overline{A} \cup \overline{B}}$ VIII. $\overline{\overline{A} \cap \overline{B}}$

a) only II.

b) only V.

c) only VIII.

d) II., V., and VII.

e) III., V., and VIII.

Solution:

Correct answer: d

5. A rectangle is partitioned into 4 subrectangles as shown below. If the subrectangles have the indicated areas, find the area of the unknown rectangle.

60	100
?	80

a) 72

b) 21

c) 70

d) 48

e) 64

Solution:

Correct answer: d

As in the figure to the right, $cb = 60$, $bd = 100$, and $ad = 80$.
Then $ac = ad \cdot \frac{c}{d} = 80 \cdot \frac{cb}{bd} = 80 \cdot \frac{60}{100} = 48$.

b	60	100
a	?	80
	c	d

6. In the expansion of $(2x+y)^{30}$, what is the coefficient of x^2y^{28}?

a) 312

b) 435

c) 392

d) 1568

e) 1740

Solution:

Correct answer: e
30 choose 2 is 435 and multiplying by 2^2 gives 1740

7. If $\cos(\theta)+\sin(\theta) = \frac{\sqrt{10}}{5}$ and $\tan(\theta) = \frac{a}{b}$, then $\frac{a}{b} + \frac{b}{a} =$

a) $\frac{5}{3}$

b) $\frac{-10}{3}$

c) $\frac{10}{3}$

d) $\frac{3}{10}$

e) 1

Solution:

Correct answer: b

If $\tan(\theta) = \frac{a}{b}$, then $\sin(\theta) = \frac{a}{\sqrt{a^2+b^2}}$ and $\cos(\theta) = \frac{b}{\sqrt{a^2+b^2}}$.

Thus $\frac{\sqrt{10}}{5} = \sin(\theta)+\cos(\theta) = \frac{a+b}{\sqrt{a^2+b^2}}$. Squaring and simplifying we obtain $3a^2+3b^2 = -10ab$. Thus $\frac{a}{b}+\frac{b}{a} = \frac{-10}{3}$.

8. Two straight roads intersect at right angles. A car and bicycle meet at the intersection each traveling on a different road. The car is going 20 miles an hour and the bicycle is traveling at one fifth the speed of the car. After 30 minutes how far apart are they?

a) 10 miles

b) $\sqrt{110}$ miles

c) $\sqrt{104}$ miles

d) 11 miles

e) 8 miles

Solution:

Correct answer: c

After 30 minutes the car is 10 miles and the bicycle is 2 miles. The hypotenuse is $x^2 = 100+4$.

9. Find the sum of all the even integers from 546 to 854 inclusive:

$$546+548+550+\cdots+852+854.$$

a) 106,400

b) 109,200

c) 107,800

d) 107,100

e) 108,500

Solution:

Correct answer: e

$$(546+854)+(548+852)+(550+850)+\cdots+(696+704)+(698+702)+700$$

$$1400+1400+1400+\cdots+1400+1400+700$$

Since $698-546 = 152$ and we have only even numbers we have $76+1 = 77$ times 1400 are added together.

$$(77\times 1400)+700 = 108500$$

10. Find the *shortest* path which starts at the origin and visits all five of the following points and returns to the origin: $\{(0,0), (1,0.5), (2,1), (2,0), (0,3)\}$.

a) $(0,0),(1,0.5),(2,1),(2,0),(0,3),(0,0)$

b) $(0,0),(2,0),(1,0.5),(2,1),(0,3),(0,0)$

c) $(0,0),(1,0.5),(2,0),(2,1),(0,3),(0,0)$

d) $(0,0),(0,3),(1,0.5),(2,0),(2,1),(0,0)$

e) $(0,0),(0,3),(2,0),(1,0.5),(2,1),(0,0)$

Solution:

Correct answer: c
By inspection the path (c) is shorter than the path (d) and the path (a) is shorter than the path (e). Let x be the distance from $(0,0)$ to $(1,0.5)$. Then the distance from $(1,0.5)$ to $(2,0)$ and the distance $(1,0.5)$ to $(2,1)$ are also both equal to x. Notice that $x > 1$. Let y be the $(2,0)$ to $(0,3)$ and z be the distance from $(2,1)$ to $(0,3)$. Then $z < y$.
The length of the path (a) is $2x + y + 4$.
The length of the path (b) is $2x + z + 5$.
The length of the path (c) is $2x + z + 4$.
Thus the path (c) is the shortest.

11. A ladder 20 feet long leans against a building. If the bottom of the ladder slides away from the building horizontally at a rate of 4 ft/sec, how fast is the ladder sliding down the house when the top of the ladder is 16 feet from the ground.

a) 3 ft/sec
b) 4 ft/sec
c) 2 ft/sec
d) 4 in/sec
e) 5 ft/sec

Solution:

Correct answer: a
When the top of the latter is 16 feet from the ground the distance from the bottom of the latter to the building is 12 ft.

$$20^2 = 12^2 + 16^2$$

Letting length of the latter be z, the distance from the bottom of the latter to the building be x, and the distance from the top of the latter to the ground be y find an equation for rates of change.

$$2x(\triangle x/\text{sec}) + 2y(\triangle y/\text{sec}) = 2z(\triangle z/\text{sec})$$

Since the length of the latter does not change we have

$$x(\triangle x/\text{sec}) + y(\triangle y/\text{sec}) = 0.$$

Plugging in for x, y and $\triangle x/\text{sec}$,

$$\triangle y/\text{sec} = -3ft/\text{sec}$$

Since the question is asking how fast the latter is sliding down the answer is 3 ft/sec.

12. Let z be a complex number of magnitude 1 such that z is not -1 or 1. Let $i = \sqrt{-1}$, and let c and d be real numbers with $c + di = \dfrac{1}{1+z}$. Which of the following is a possible value for c?

a) $\frac{1}{2}$
b) $\frac{1}{\sqrt{3}}$
c) 1
d) $\frac{-1}{2}$
e) $\frac{-1}{\sqrt{3}}$

Solution:

Correct answer: a

Let $z = a + bi$. Then $\frac{1}{1+z} = \frac{1}{1+z} \cdot \frac{1+\overline{z}}{1+\overline{z}} = \frac{(1+a)-bi}{2(1+a)}$. Thus $c = \frac{1}{2}$.

13. The number of real solutions to $\frac{x}{100} = \sin(x)$ is which of the following?

a) 32 b) 56 c) 63

d) 64 e) 65

Solution:

Correct answer: c

On the interval $[0,\pi]$, sin increases 0 to 1 and then decreases back to 0. On the interval $[\pi, 2\pi]$, sin decreases from 0 to -1 and then increases back to 0. This repeats because sin is periodic with period 2π. The graph of $y = x/100$ is a straight line. This line intersects the graph of $y = \sin x$ twice on the interval $[0,\pi]$, $[2\pi, 3\pi]$, ... as long as the right endpoint is less than 100. Notice that $31\pi < 100$ and $32\pi > 100$. So for $x \geq 0$, there are 32 solutions. Similarly for $x \leq 0$, there are 32 solutions. The solution $x = 0$ was counted twice so there are 63 solutions.

14. Convert 6023.22 from base 7 to a decimal (base 10). What is the digit in the tens position?

a) 0 b) 1 c) 3

d) 5 e) 7

Solution:

Correct answer: e

In decimal the number is $2075 + \frac{16}{49}$.

15. Let n be greatest integer that will divide the three integers 13511, 13903 and 14589 and leave the same remainder. Which is the sum of the digits of n?

a) 7 b) 9 c) 13

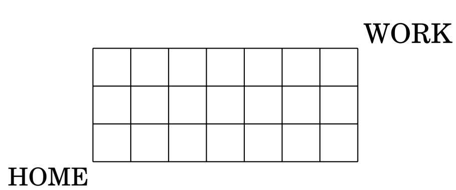
d) 17 e) 24

Solution:

Correct answer: d

The differences of the numbers are 1078, 686, and 392. n must divide each of these numbers. Factoring them gives $(2)(7)(7)(11)$, $(2)7^3$, and $2^3 7^2$. Thus $n = 98$.

16. A young man has to walk from his home to his work. (His mom picks him up so he does not walk home.) How many days can he walk to work in a different way by walking a total of three blocks north and seven blocks east in any order? (No cutting diagonal through the blocks).



a) 1 b) 10 c) 80

d) 120 e) 100

Solution:

Correct answer: d
The young man needs to walk a total of 10 blocks. 3 of the blocks need to be north and the remaining east. ${}_{10}C_3 = 120$

17. Find the largest integer n such that n has exactly 4 positive divisors and n divides 100!.

a) 716539 b) 8633 c) 5293

d) 29791 e) 100

Solution:

Correct answer: d
To have exactly 4 positive divisors a number needs to be the product of 2 distinct prime numbers or a prime cubed. Since 100! = (100)(99)(98).....(5)(4)(3)(2)(1) if you choose the largest two prime which are less than 100 you get (97)(89) = 8633. The largest prime whose cube divides 100! is 31 and $31^3 = 29791$.

18. Consider the checkerboard below, with some of the squares removed as signified by the X's. How many ways are there to place 5 circles on the remaining squares (those without X's) so that no two circles are on the same row or column?

				X
			X	
		X		
	X			
X				

a) 44 b) 120 c) 40

d) 96 e) 45

Solution:

Correct answer: a

First let us label the columns and rows.

	1	2	3	4	5
A					X
B				X	
C			X		
D		X			
E	X				

Now make a large table of combinations which do not put a circle on an X.

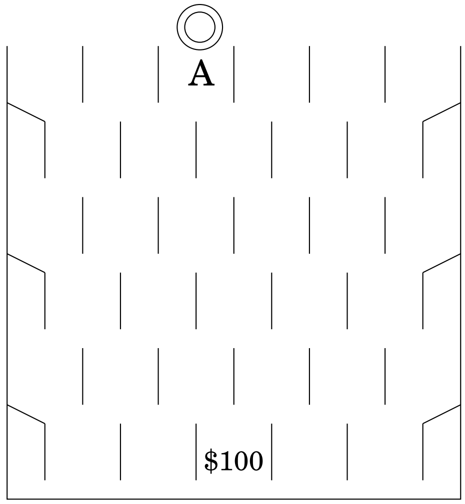
1	2	3	4	5
A	B	D	C	E
			E	C
		E	C	D
			D	C
	C	B	D	E
			E	D
		D	E	B
		E	D	B
	E	B	C	D
			D	C
		D	C	B

1	2	3	4	5
B	A	D	C	E
			E	C
		E	C	D
			D	C
	C	A	D	E
			E	D
		D	A	E
		E	A	D
	E	A	C	D
			D	C
		D	A	C

1	2	3	4	5
C	A	B	D	E
			E	D
		D	E	B
		E	D	B
	B	A	D	E
			E	D
		D	A	E
		E	A	D
	E	A	D	B
		B	A	D
		D	A	B

1	2	3	4	5
D	A	B	C	E
			E	C
		E	C	B
	B	A	C	E
			E	C
		E	A	C
	C	A	E	B
		B	A	E
		E	A	B
	E	A	C	B
		B	A	C

19. Elizabeth is a contestant on the *Price is Right* and her game is **PLINKO**. She has one token to drop from the top. On the game board below Elizabeth chooses slot **A** to drop her token into. At each divider there is an equal chance of going left and right. What is the probability Elizabeth will get the prize of $100?



a) $\frac{1}{5}$

b) $\frac{3}{16}$

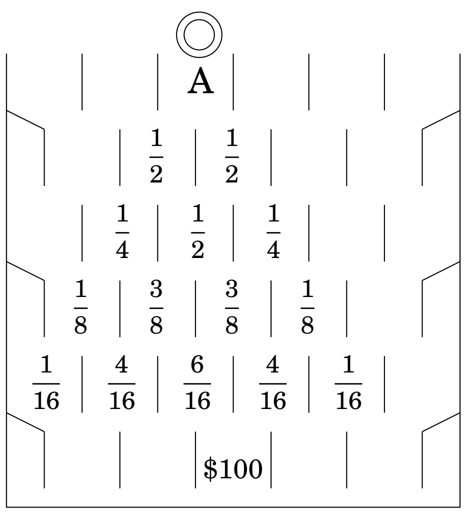
c) $\frac{8}{32}$

d) $\frac{10}{31}$

e) $\frac{5}{16}$

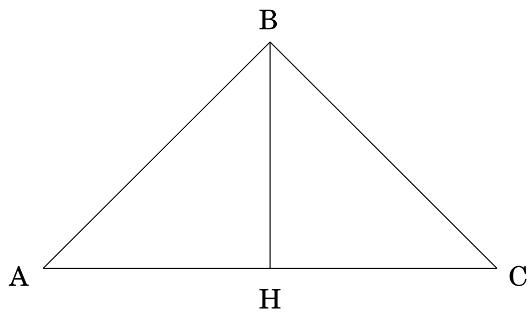
Solution:

Correct answer: e



Thus the probability is $\frac{1}{2}\left(\frac{6}{16}+\frac{4}{16}\right)=\frac{5}{16}$.

20. The line BH is perpendicular to line AC and H is on the line AC. The angle BAC and BCA both measure $\frac{\pi}{6}$. If BH is length 4, what is the length of AC?
Diagram is may not be drawn to scale.



a) $8\sqrt{3}$

b) $4\frac{\sqrt{3}}{2}$

c) 16

d) 10

e) $4\frac{\sqrt{2}}{2}$

Solution:

Correct answer: a
The triangle $\triangle ABH$ is congruent to the triangle $\triangle BCH$.
The angle $\angle BAH$ is complementary to the angle $\angle ABH$ since $\triangle ABH$ is a right triangle.
Thus $\triangle ABH$ is a $\left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ right triangle and AH has length $4\sqrt{3}$. So the length of AC is $8\sqrt{3}$.

21. Let $f(x)$ be an odd function defined on $\mathbb{R}$ satisfying

(a) $f(x+2) = -f(x)$, for all real numbers x;
(b) $f(x) = 2x$ when $0 \le x \le 1$.

Find the value of $f(10\sqrt{3})$.

a) $20\sqrt{3} - 36$

b) $-20\sqrt{3} + 36$

c) $20\sqrt{3} - 32$

d) $10\sqrt{3} + 18$

e) $-20\sqrt{3} - 36$

Solution:

Correct answer: b
Since $f(x+2) = -f(x)$, we have

$$f(x+4) = -f(x+2) = f(x).$$

Hence f is a periodic function with period 4. Then using that $f(x)$ is an odd function and $f(x) = 2x$ when $0 \le x \le 1$, we have $\begin{aligned} f(10\sqrt{3}) &= f(10\sqrt{3}-16) = -f(-10\sqrt{3}+16) \\ &= f(-10\sqrt{3}+18) = 2(-10\sqrt{3}+18) = -20\sqrt{3}+36 \end{aligned}$, since $0 < -10\sqrt{3} + 18 < 1$.

22. How many four-digit integers either leave a remainder of 2 when divided by 7, or a remainder of 4 when divided by 5, but not both?

a) 2570

b) 2572

c) 2575

d) 3084

e) 3086

Solution:

Correct answer: b

Let A be the number of four-digit integers that leave a remainder of 2 when divided by 7.

Let B be the number of four-digit integers that leave a remainder of 4 when divided by 5.

Let C be the number of four-digit integers that leave a remainder of 2 when divided by 7 and a remainder of 4 when divided by 5.

Our goal is to find $(A-C)+(B-C)$. <u>Find A</u> Looking for integers x such that

$$1000 \le 7x+2 < 10000$$

Subtract 2 and divid by 7.

$$142+\frac{4}{7} \le x < 1428+\frac{2}{7}$$

Since x is an integer.

$$143 \le x \le 1428$$

$A = 1428-143+1 = 1286.$

<u>Find B</u> Similar calculations with y

$$1000 \le 5y+4 < 10000$$

$$200 \le y \le 1999$$

$B = 1800.$

<u>Find C</u> Similar calculations with z

$$1000 \le 35z+9 < 10000$$

$$29 \le z \le 285$$

$C = 257$

$$(A-C)+(B-C) = 2572$$

23. Let a_n and b_n be sequences of numbers satisfying

$$a_1 = -1,\ b_1 = 2,\ a_{n+1} = -b_n,\ b_{n+1} = 2a_n - 3b_n \text{ for all natural number } n.$$

Compute $b_{2017}+b_{2016}$.

a) 3×2^{2015} b) 3×2^{2017} c) 3×2^{2014}

d) 3×2^{2016} e) 5×2^{2016}

Solution:

Correct answer: d
Since $a_{n+1} = -b_n$, $b_{n+2} = 2a_{n+1} - 3b_{n+1}$, we have

$$b_{n+2} + b_{n+1} = -2(b_{n+1} + b_n) = (-2)^2(b_n + b_{n-1}) = \cdots = (-2)^n(b_2 + b_1),$$

which implies

$$b_{2017} + b_{2016} = (-2)^{2015}(2-8) = 3 \times 2^{2016}.$$

24. Find all real numbers a such that $f(x) = x^2 + a|x-1|$ is an increasing function on the interval $[0,\infty)$.

a) $[-2,\infty)$
b) $[-2,0]$
c) $(-\infty,0]$
d) $[0,2]$
e) $[-2,2]$

Solution:

Correct answer: b
For $0 \le x \le 1$, $f(x) = x^2 - ax + a$. Thus $f(x)$ is increasing on the interval $[0,1]$ if and only if $a \le 0$. On the interval $[1,\infty)$, $f(x) = x^2 + ax - a$. Hence, $f(x) = x^2 + ax - a$ is increasing on $[1,\infty)$ if and only if $a \ge -2$. Hence $[-2,0]$ is the set of real numbers such that the function is increasing.

25. Find the area of triangle ΔABC if $AB = AC = 50$ in and $BC = 60$ in.

a) 2000 square inches
b) 1500 square inches
c) 1000 square inches
d) 2400 square inches
e) 1200 square inches

Solution:

Correct answer: e
Since the triangle is isosceles , the altitude AX to the side BC bisect BC. Hence, using the Pythagorean Theorem on the right triangle ABX, we have $AX = 40$ in. Therefore, the area is $(BC)(AX)/2 = 60 \times 40/2$ square inches $= 1200$ square inches.

26. A group of airplanes is based on a small island. The tank of each plane holds just enough fuel to take it halfway around the world. Any desired amount of fuel can be transferred from the tank of one plane to the tank of another while the planes are in flight. The only source of fuel is on the island. It is assumed that there is no time lost in refueling either in the air or on the ground. The planes have the same constant speed and rate of fuel consumption. What is the smallest number of planes that will ensure the flight of one plane around the world on a great circle and have all the planes return safely to their island base?

a) 3
b) 4
c) 6
d) 7
e) 9

Solution:

Correct answer: a

Planes A, B, and C take off together. After going $\frac{1}{8}$ of the distance, C transfers $\frac{1}{4}$ tank to A and $\frac{1}{4}$ to B and C returns. Planes A and B continue another $\frac{1}{8}$ of the way, then B transfers $\frac{1}{4}$ of a tank to A and B returns.
A flies $\frac{3}{4}$ the way around and is met by B who transfers $\frac{1}{4}$ of a tank to A. B turns around and follows A to the base. At $\frac{1}{8}$ the way from the base they are met by C who transfers $\frac{1}{4}$ of a tank to each and they all return to the base safely.

27. Find the units digit of 13^{2017}.

a) 5 b) 7 c) 9

d) 1 e) 3

Solution:

Correct answer: e
The units digit of 13^{2017} is the same as that of 3^{2017}. Since $2017 = 4 \times 504 + 1$, we have

$$3^{2017} = 3^{4\times504+1} = 3^{4\times504}3 = (3^4)^{504}3 = 81^{405}3.$$

So the units digit is 3.

28. Let $f(x) = \ln\left(\frac{1+x}{1-x}\right)$. What is $f\left(\frac{3x+x^3}{1+3x^2}\right)$ in terms of $f(x)$?

a) $f(x)$ b) $-3f(x)-e$ c) $\frac{f(x)+1}{f(x)-1}$

d) $f(x)^2 + f(x)$ e) $3f(x)$

Solution:

Correct answer: e

$$f\left(\frac{3x+x^3}{1+3x^2}\right) = \ln\left(\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}}\right) = \ln\left(\frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3}\right) = \ln\left(\frac{(1+x)^3}{(1-x)^3}\right) = 3f(x)$$

29. Amy likes either red or green clothes, but not both. She likes either turtleneck or V-neck sweaters, but not both. Amy never wears a sweater that is both the color and the type she likes, nor does she wear one that is neither the color nor the type she likes. Amy wears a red turtleneck. If you want to buy a sweater for Amy that she will wear, should you buy a red V-neck, a green V-neck, or a green turtleneck?

a) A red V-neck b) A green V-neck c) A green turtle neck

d) Amy won't wear any of those sweaters e) Not enough information is given to answer the question

Solution:

Correct answer: b
She likes exactly two of the four options, {red, green, V-neck, turtleneck}. She also likes exactly one of the options {red, turtleneck}. Thus she must like exactly one of the options {green, V-neck}.

30. The parabolas $y = x^2$ and $y = -x^2 + 4x - \frac{5}{2}$ have two common tangent lines. The one with smaller slope intersects the x-axis somewhere between $x = 0$ and $x = \frac{3}{5}$. Where?

a) At $x = 0$ b) At $x = \frac{1}{5}$ c) At $x = \frac{1}{4}$

d) At $x = \frac{1}{3}$ e) At $x = \frac{1}{2}$

Solution:

Correct answer: c
The common tangent line is the line with slope 1 through the points $(1/2, 1/4)$ and $(3/2, 5/4)$. (The other common tangent has slope 3 and intersects x-axis at $\frac{3}{4}$.)

31. Let $\frac{113}{50} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \frac{1}{e}}}}$, where a, b, c, d and e are integers. Find c and e.

a) $c = 1$ and $e = 2$ b) $c = 3$ and $e = 1$ c) $c = 2$ and $e = 3$

d) $c = 2$ and $e = 4$ e) $c = 3$ and $e = 5$

Solution:

Correct answer: a
$(a, b, c, d, e) = (2, 3, 1, 5, 2)$ by calculation.

32. Find the greatest common divisor of 123432123432100 and 2468642468642000015.

a) 1 b) 3 c) 5

d) 7 e) 15

Solution:

Correct answer: c
For $n = 123432123432100$ and $m = 2468642468642000015$, we have $n = 20000m + 15$ So a common divisor of n and m must divide 15. Both n and m is divisible by 5 since their last digit is 0 or 5. The number n is not divisible by 3 since the sum of the digits of n is not divisible by 3. Thus the greatest common divisor is 5

33. Consider the sequence $x_0 = 1, x_1 = 2, x_2 = \frac{5}{2}, x_3 = \frac{11}{4}, x_4 = \frac{23}{8}, \ldots$. The pattern is $x_n = \frac{3x_{n-1} - x_{n-2}}{2}$ for $n > 1$. Find the limit of these numbers.

a) 2

b) 2.5

c) 2.9

d) 3

e) 3.4

Solution:

Correct answer: d

Numbers are $x_n = \dfrac{3 \cdot 2^{n-1} - 1}{2^{n-1}}$.

34. Find the area of overlap between the closed disc $x^2 + y^2 \le 2$ and the parabolic region $y \ge x^2$.

a) $\dfrac{\pi}{3} + \dfrac{\sqrt{2}}{2}$

b) 2.5.

c) $\dfrac{\pi}{6} + 3\sqrt{2}$

d) 2.

e) $\dfrac{\pi}{2} + \dfrac{1}{3}$

Solution:

Correct answer: e

Equal to one-fourth of circle of radius $\sqrt{2}$ plus twice the area between $y = x$ and $y = x^2$.

35. How many pairs of integers (x, y) satisfy the following equation.

$$x^3 + 6x^2 + 8x = 3y^2 + 9y + 1.$$

a) 0

b) 1

c) 2

d) 3

e) None of the above

Solution:

Correct answer: a

Notice that $x^3 + 6x^2 + 8x = x(x^2 + 6x + 8) = x(x+4)(x+2)$ which must be divisible by three because either $x, x+2, x+4$ is divisible by 3. However, $3y^2 + 9y + 1$ has a remainder of 1 when divided by 3. Thus there 0 pairs of integers satisfying the equation.

36. Find the shortest distance from the point $(16, 1/2)$ to the parabola $y = x^2$ in the plane.

a) $\dfrac{7}{2}\sqrt{17}$ units

b) 14 units

c) 15 units

d) $10\sqrt{2}$ units

e) 10 units

Solution:

Correct answer: a

Line from point $(16, 1/2)$ to (x, x^2) must have slope $\dfrac{-1}{2x}$. Thus the equation $2x^3 = 16$ with solution $x = 2$. Distance from $(2, 4)$ to $(16, 1/2)$ is the answer.

37. $\sqrt[3]{2\sqrt{13} + 5} - \sqrt[3]{2\sqrt{13} - 5} =$

a) -1

b) $2\sqrt{13}$

c) 10

d) 1

e) 3

Solution:

Correct answer: d

Let $A = \sqrt[3]{2\sqrt{13}+5}$ and $B = \sqrt[3]{2\sqrt{13}-5}$. Then we wish to solve for $A-B$. Notice that $A^3 - B^3 = 2\sqrt{13}+5-(2\sqrt{13}-5) = 10$ and $AB = \sqrt[3]{(2\sqrt{13}+5)(2\sqrt{13}-5)} = 3$.
$(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3 = A^3 - B^3 - 3AB(A-B) = 10 - 9(A-B)$
$(A-B)^3 + 9(A-B) - 10 = \big((A-B)-1\big)\big((A-B)^2 + (A-B) + 10\big) = 0$. Since $A-B$ is real, $A-B = 1$.

38. Let S be a subset of the set $\{1,6,16,30,57,113,233,465,931,1856,3717,7432,14865,29731,59454\}$ so that the elements in S add up 35573. Find the two smallest elements of S.

a) 1 and 3717

b) 233 and 1856

c) 113 and 233

d) 6 and 931

e) 6 and 30

Solution:

Correct answer: e

Each term in the sequence is greater than the sum of all the terms before it (superincreasing sequence). So, if 35573 is a sum, the largest term in the sum must be 29731. The difference between 35573 and 29731 is 5842. So the next largest term must be 3717. The difference between 5842 and 3717 is 2125, so the next largest term must be 1856. The difference between 2125 and 1856 is 269, so the next largest term must be 233. The difference between 269 and 233 is 36 so the last two terms must be 6 and 36.

39. Suppose real numbers x, y, z satisfy the equation $\dfrac{x}{y+z} + \dfrac{y}{z+x} + \dfrac{z}{x+y} = 1$.

Compute the value of

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}.$$

a) 1

b) -1

c) 2

d) 0

e) -2

Solution:

Correct answer: d

We first note that $x+y+z \neq 0$. Otherwise,

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=-3.$$

Thus, we have

$$\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)(x+y+z)=(x+y+z).$$

Simplifying it, we have

$$\frac{x^2}{y+z}+x+\frac{y^2}{z+x}+y+\frac{z^2}{x+y}+z=(x+y+z),$$

which yields

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}=0.$$

40. Let $f(x)=4\sin^3 x-\sin x+2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2$. What is the minimal period of $f(x)$?

a) 2π

b) $\pi/2$

c) $2\pi/3$

d) 2π

e) π

Solution:

Correct answer: c

Using the triangle identities, we have

$$\begin{aligned}
f(x) &= 4\sin^3 x-\sin x+2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2\\
&= 4\sin^3 x-\sin x+2\left(1-2\sin\frac{x}{2}\cos\frac{x}{2}\right)\\
&= 4\sin^3 x-3\sin x+2\\
&= \sin x(4\sin^2 x-3)+2\\
&= \sin x(3(\sin^2 x-1)+\sin^2 x)+2\\
&= \sin x(-3\cos^2 x+\sin^2 x)+2\\
&= \sin x(-2\cos^2 x-(\cos^2 x-\sin^2 x))+2\\
&= -\sin 2x\cos x-\sin x\cos 2x+2\\
&= -\sin 3x+2.
\end{aligned}$$

The answer is $2\pi/3$.

END OF EXAM