State Math Contest 2015 - Senior Exam Solutions

1. If $4^x - 2^x - 6 = 0$, then $x =$

   a) $\log_2 3$ b) -2 c) $\log_4 3$ d) 3 e) $\log_3 2$

   Solution: Let $y = 2^x$. Then the equation is $y^2 - y - 6 = 0$, or $(y-3)(y+2) = 0$, which means that $y = 3, -2$. It is not possible for $-2 = 2^x$, so we have $2^x = 3$ or $x = \log_2 3$.

   The answer is **a**.

2. The sum of the infinite series $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots$ equals

   a) 1 b) $\frac{9}{10}$ c) $\frac{4}{5}$ d) $\frac{7}{10}$ e) $\frac{3}{4}$

   Solution: $1 + \frac{-1}{4} + \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^3 + \cdots$ This is a geometric series whose sum is $\frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{4}{5}$.

   The answer is **c**

3. How many triplets (x, y, z) satisfy the following three equations?

$$3x + 2y + 4z = 5,$$
$$x - 2y - 3z = 4,$$
$$6x + 4y + 8z = 10.$$

   a) 1 b) 2 c) infinitely many

   d) 0 e) a finite number, but more than 2.

   Solution: $x = 4 + \frac{7+13z}{4} - 3z, y = \frac{-7-13z}{8}, z$, where z is arbitrary.

   The answer is **c**.

4. There are 60 students in a class. Of the students, 36 are in club A, 26 students are in club B. The number of students that are in neither of the clubs is four more than one third of the number of students that are in both clubs. Find the number of students in both clubs.

   a) 5 b) 6 c) 7 d) 8 e) 9

   Solution: Let n be the number of students in neither club and b be the number of students in both clubs. $n = 4 + \frac{b}{3}$ and $n + 36 + 26 - b = 60$. Solving for b gives 9.

   The answer is **e**.

5. The probability that it will rain at some point during 7:00-8:00 PM is 80%. The probability that it will rain at some point during 8:00 - 9:00 PM is also 80%. If these two probabilities are independent, what is the probability that it will rain at some point during 7:00 -9:00 PM?

   a) 64 % b) 96 % c) 80 % d) 20 % e) 16 %

   Solution: The probability it will not rain from 7-8 is .2 and the probability it will not rain from 8-9 is .2. Since they are independent for both to happen the probability is .04. Thus the probability that it does happen is 1-.04 or .96, or 96%.

   The answer is **b**.

6. A polynomial of degree 4 whose coefficients are real numbers has zeros 1, 2, and $4-i$. If the coefficient of x^4 is 1, what is the coefficient of x^3?

   a) -3 b) -8 c) -11 d) 19 e) 7

   Solution: $(x-1)(x-2)(x-4+i)(x-4-i) = (x^2-3x+2)(x^2-8x=17) = x^4-11x^3+43x^2-67x+34$

   The answer is **c**.

7. Find $\arctan 2 + \arctan 3$ in degrees.

   a) 120 b) 135 c) 150 d) 180 e) 315

   Solution: Let $x = \arctan 2 + \arctan 3$. Then the tangent-sum formula gives:

$$\tan x = \frac{\tan(\arctan 2) + \tan(\arctan 3)}{1 - \tan(\arctan 2)\tan(\arctan 3)} = \frac{5}{1-6} = -1$$

   There are two places where $\tan x = -1$: $135°$, and $325°$. $45° = \arctan 1 < \arctan 2 < \arctan 3 < 90°$. So $90° < \arctan 2 + \arctan 3 < 180°$. Since $\tan(\arctan 2 + \arctan 3) = -1$, then $\arctan 2 + \arctan 3 = 135°$.

   The answer is **b**.

8. A coin is flipped 8 times. What is the probability that the number of heads is strictly greater than the number of tails?

   a) $\frac{4}{9}$ b) $\frac{1}{2}$ c) $\frac{93}{256}$ d) $\frac{23}{84}$ e) $\frac{61}{84}$

   Solution: First, we calculate the probability that we flip 4 heads and 4 tails:

$$P(4H, 4T) = \frac{\binom{8}{4}}{2^8} = \frac{\frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2\cdot1}}{256} = \frac{70}{256}$$

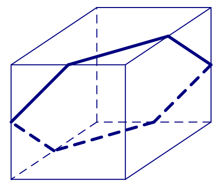
   So, the probability that we do *not* roll the same number of heads as tails is $\frac{186}{256}$. Once we rule out the above, the probability of rolling more heads than tails is exactly the same as rolling more tails than heads, so we just need to divide this number by 2. We get $\frac{93}{256}$.

   The answer is **c**.

9. A plane slices a cube so that the intersection is a polygonal region. What is the maximum number of sides of the polygonal region?

a) 3 b) 4 c) 5 d) 6 e) 7

Solution: The intersection of a plane with the face of the cube is a line segment. So the polygonal region can have at most six sides because a cube has six faces. The following picture shows that six is possible.



The answer is **d**.

10. The four consecutive coordinates of a rhombus in the coordinate plane are (a, b), (c, d), (e, f), and (g, h), where $a \neq e$ and $c \neq g$. If $\log \dfrac{b-f}{a-e} = 1$, then $\log \dfrac{d-h}{g-c} =$

a) -10 b) -1 c) $-\dfrac{1}{10}$ d) $\dfrac{1}{10}$ e) 10

Solution: Given a rhombus as described in the problem, we know that the line connecting (a, b) to (e, f) must be perpendicular to the line connecting (c, d) to (g, h).
Notice that $\dfrac{a-e}{b-f}$ is the slope of the first line. So if $\log \dfrac{a-e}{b-f} = 1$, then the slope of this line must be 10. This means that the slope of the second line must be $m = -\dfrac{1}{10}$, since the two are perpendicular. $\dfrac{d-h}{g-c} = -m = \dfrac{1}{10}$ (since the order of the terms in the numerator doesn't match the order in the denominator). So, $\log \dfrac{d-h}{g-c} = -1$.

The answer is **b**.

11. I have letters to four people, and envelopes addressed to the four people, and (without looking) I randomly put a letter into each of the four envelopes. What is the probability that none of the letters will get put into its correct envelope?

a) $\dfrac{1}{4}$ b) $\dfrac{3}{4}$ c) $\dfrac{81}{256}$ d) $\dfrac{3}{8}$ e) $\dfrac{23}{24}$

Solution: First I count the total number of ways in which I can put letters in envelopes: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Next, Suppose that the envelopes are for person # 1, # 2, # 3, and # 4. Then in how many ways can I get everyone's letters wrong? The following table lists all possible ways to do this:

Envelopes			
1	2	3	4
2	1	4	3
2	3	4	1
2	4	1	3
3	1	4	2
3	4	1	2
3	4	2	2
4	1	2	3
4	3	1	2
4	3	2	1

So, the probability is $\frac{9}{24} = \frac{3}{8}$.

The answer is **d**.

12. How many numbers in the range from 1 to 1,000, inclusive, are not divisible by the first three primes (2, 3 or 5)?

    a) 166 b) 266 c) 299 d) 701 e) 734

    Solution: Let's count how many numbers between 1 and 1,000 (inclusive) *are* divisible by 2, 3, or 5.
    How many numbers are divisible by 2?
    $2, 4, 6, \ldots, 1,000$: 500
    How many numbers are divisible by 3?
    $3, 6, 9, \ldots, 999$: 333
    How many numbers are divisible by 5?
    $5, 10, 15 \ldots, 1,000$: 200
    If we add these together, we get 1,033. But we have over counted! Let's subtract the following:

    How many numbers are divisible by 2 and 3? (We counted them twice!)
    $6, 12, 18, \ldots, 996$: 166
    How many numbers are divisible by 2 and 5? (We counted them twice!)
    $10, 20, 30, \ldots, 100$: 100
    How many numbers are divisible by 3 and 5? (We counted them twice!)
    $15, 30, 45, \ldots, 990$: 66
    So, now we have $1,033 - 166 - 100 - 66 = 701$.

    But what about the numbers that are divisible by 2, 3, and 5? We counted them three times in the first step. But then we subtracted them three times in the second step. We need to add them back in.
    How many numbers are divisible by 2, 3 and 5?
    $30, 60, \ldots, 990$: 33
    So, we get 734 numbers which are divisible by at least one of 2, 3, or 5. Which means there are 266 numbers which are not divisible by 2, 3, or 5.

    The answer is **b**.

13. If a function f satisfies $f(n^2) = 2f(n) + 1$ and $f(2) = 3$, then $f(256)$ equals

a) 6 b) 17 c) 24 d) 27 e) 31

Solution: $f(256) = f(16^2) = 2f(16) + 1 = 2f(4^2) + 1 = 2(2f(4) + 1) + 1 = 2(2f(2^2) + 1) + 1 = 2(2(2f(2) + 1) + 1) + 1 = 8f(2) + 4 + 2 + 1 = 24 + 7 = 31$.

The answer is **e**.

14. If x is in the domain of the function $f(x) = \dfrac{(x^2 - 1)\sqrt{(3x - 2)}}{x - 1}$, then

a) $0 < x < 2$ or $2 < x < \infty$

b) $-\infty < x < 1$ or $1 < x < \infty$

c) $\dfrac{2}{3} \leq x$

d) $\dfrac{2}{3} \leq x < 1$ or $1 < x < \infty$

e) x is any real number.

Solution: $3x - 2 \geq 0$ and $x \neq 1$ so $\left[\dfrac{2}{3}, 1\right)$ or $(1, \infty)$.

The answer is **d**.

15. The expression $\dfrac{\sqrt{x + 4} - \sqrt{x}}{4}$ is the same as

a) $\dfrac{1}{\sqrt{x + 4} + \sqrt{x}}$ b) $\dfrac{4}{\sqrt{x + 4} + \sqrt{x}}$ c) $\dfrac{1}{2}$

d) $\dfrac{1}{\sqrt{x + 4} - \sqrt{x}}$ e) $\dfrac{2x + 4}{\sqrt{x + 4} - \sqrt{x}}$

Solution: $\dfrac{\sqrt{x + 4} - \sqrt{x}}{4} = \dfrac{(\sqrt{x + 4} - \sqrt{x})(\sqrt{x + 4} + \sqrt{x})}{4(\sqrt{x + 4} + \sqrt{x})} = \dfrac{x + 4 - x}{4\sqrt{x + 4} + \sqrt{x}} = \dfrac{1}{\sqrt{x + 4} + \sqrt{x}}$

The answer is **a**.

16. If we are to choose real numbers a and b such that $0 < a < b$, then we can best describe the solution set of $|x + 12| < |3x - 1|$ as the set of all x such that

a) $-a < x < b$ b) $-b < x < a$

c) $x < -a$ or $x > b$ d) $x < -b$ or $x > a$

e) $x < a$ or $x > b$

Solution: **Case 1** $x + 12 \geq 0$ and $3x - 1 \geq 0$ so $x \geq -12$ and $x \geq \dfrac{1}{3}$. This means $x \geq \dfrac{1}{3}$. Also $x + 12 < 3x - 1$ so $\dfrac{13}{2} < x$. To satisfy both conditions $\dfrac{13}{2} < x$.

Case 2 $x + 12 \geq 0$ and $3x - 1 < 0$ so $x \geq -12$ and $x < \dfrac{1}{3}$. Also $x + 12 < -3x + 1$ so $x < \dfrac{-11}{4}$. Both conditions give $-12 \leq x < \dfrac{-11}{4}$.

Case 3 $x+12<0$ and $3x-1\geq 0$ so $x<-12$ and $x\geq \frac{1}{3}$ which is impossible.

Case 4 $x+12<0$ and $3x-1<0$ so $x<-12$ and $x<\frac{1}{3}$ or $x<-12$. Also $-x-12<-3x+1$ so $x<\frac{13}{2}$. Both conditions give $x<-12$

Combining all for cases gives $\frac{13}{2}<x \cup -12\leq x<\frac{-11}{4}\cup x<-12$. Let $a=\frac{11}{4}$ and $b=\frac{13}{2}$ then $b<x$ or $-a>x$.

The answer is **c**.

17. Find the inverse function $f^{-1}(x)$ if $f(x)=\frac{2x-1}{x+3}$.

a) $f^{-1}(x)=\frac{3x-1}{2-x}$ b) $f^{-1}(x)=\frac{3x+1}{2-x}$

c) $f^{-1}(x)=\frac{2x+1}{3-x}$ d) $f^{-1}(x)=\frac{x+1}{2+2x}$

e) $f^{-1}(x)=\frac{2-x}{1+3x}$

Solution: $y=\frac{2x-1}{x+3}$, $(x+3)y=2x-1$, $xy+3y-2x=-1$, $x(y-2)=-1-3y$, $x=\frac{-1-3y}{y-2}$. Finally, we switch x and y

$$y=\frac{-1-3x}{x-2}$$

The answer is **b**.

18. Train stations A and B are on the same railroad line and are 50 miles away from each other. A train leaves station A heading towards station B at 1:00 pm going 20 miles an hour. Another train leaves station B heading toward station A at 2:00 pm going 10 miles an hour. When will the two trains meet each other?

a) 2:30 pm b) 3 pm c) 3:30 pm

d) 4 pm e) 4:30 pm

Solution: Train A travels $y=20t$. Train B travels $x=10(t-1)$. They meet when $y=50-x$. So $20t=50-10(t-1)=-10t+60$. Giving $t=2$. The trains meet at 3pm.

The answer is **b**.

19. If $x^a x^b=1$ and $x>1$, find $4a-b^2+a^2+4b-10$.

a) -20 b) -10 c) 0 d) 10 e) 20

Solution: $x^{a+b}=1$ since $x\neq \pm 1$ then $a+b=0$. Since $a=-b$, -10 is the answer.

The answer is **b**.

20. Find the number of digits in the product $25^{25}\times 2^{60}$.

a) 76 b) 37 c) 54 d) 28 e) 65

Solution: $25^{25}\times 2^{60}=(5^2)^{25}\times 2^{50}\times 2^{10}=5^{50}\times 2^{50}\times 32^2=10^{50}\times 1024=0.1024\times 10^{54}$

The answer is **c**.

21. Positive integers m and n satisfy the equation $(2m - 7)(2n - 7) = 25$. What are all possible values for $m + n$?

a) 2, 20, 24 b) 3, 12, 16 c) 2, 10, 16

d) 12, 20 e) 2, 12, 20

Solution: Factors of 25 are $\pm 1, \pm 25$ or $\pm 5, \pm 5$. Consider each set of factors. For factors 1 and 25, $2m - 7 = 1$ and $2n - 7 = 25$ so $m = 4$ and $n = 16$. For factors 5 and 5, $2m - 7 = 5$ so $m = n = 6$. For factors -1, -25, $2m - 7 = -1$ and $2n - 7 = -25$ so $m = 3$ and $n = -9$, but n cannot be negative. For factors -5 and -5, $2m - 7 = -5$ so $m = n = 1$.

The answer is **e**.

22. Three sides of a quadrilateral have lengths 3, 4, and 9. There exist positive real numbers a and b such that if l is the length of the fourth side of the quadrilateral, then $a < l < b$, and if l satisfies $a < l < b$, then there exists a quadrilateral with side lengths 3, 4, 9, and l. Find $a + b$.

a) 18 b) 16 c) 9 d) 32 e) 24

Solution: If the three sides are colinear and do not overlap except at the end points they would add up to 16. If the sides are colinear and overlapped the gap between the shorter sides would be 2. So $a + b = 18$

The answer is **a**.

23. Two square regions A and B each have area 8. One vertex of square B is the center point of square A. Find the area of $A \cup B$.

a) 16 b) 15 c) $10\sqrt{2}$

d) 14 e) Cannot be determined.

Solution: Let $C = A \cap B$ be the region that lies in both A and B. By rotating C about the center point of square A by 90°, 180°, and 270° we get a tiling of A by four congruent regions. So the area of C is $2 = \frac{1}{4}$ the area of A. The area of $A \cup B = 8 + 8 - 2 = 14$.

The answer is **d**.

24. A plane slices a cone parallel to the base and one-third the distance from the vertex to the base and a second parallel plane slices the cone two-thirds the distance from the vertex to the base. What fraction of the volume of the cone is between the two slices?

a) $\frac{8}{9}$ b) $\frac{7}{27}$ c) $\frac{1}{3}$ d) $\frac{1}{9}$ e) $\frac{2}{27}$

Solution: The part of the cone above the first plane is similar to the the cone with a scaling factor of $\frac{1}{3}$ and its fraction of the volume of the cone is $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$. The part of the cone above the second plane is similar to the the cone with a scaling factor of $\frac{2}{3}$ and its fraction of the volume of the cone is $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$. So the fraction of the volume of the cone between the two slices is $\frac{8}{27} - \frac{1}{27} = \frac{7}{27}$.

The answer is **b**.

25. Suppose that the numbers 4-9, inclusive, are arranged in three pairs of distinct numbers. The numbers in each pair are added together, and the resulting three numbers are then multiplied together. What is the maximum value of the resulting product?

a) less than 1,801
b) between 1,801 and 1,900, inclusive
c) between 1,901 and 2,000, inclusive
d) between 2,001 and 2,100, inclusive
e) more than 2,100

Solution: If a, b, and c are positive numbers their arithmetic mean is $\dfrac{a+b+c}{3}$ and their geometric mean is $\sqrt[3]{abc}$. The geometric mean is always less than or equal to the arithmetic mean; i.e., $\sqrt[3]{abc} \leq \dfrac{a+b+c}{3}$ or $abc \leq \left(\dfrac{a+b+c}{3}\right)^3$. Suppose the numbers $4-9$, inclusive, are arranged in three pairs of distinct numbers. Let a, b, and c the sum of each pair. Then $a+b+c = 4+5+6+7+8+9 = 39$. So $abc \leq \left(\dfrac{a+b+c}{3}\right)^3 = \left(\dfrac{39}{3}\right)^3 = 13^3$. Since $4+9 = 5+8 = 6+7 = 13$, we know that it is possible to attain 13^3.

The answer is **e**.

26. Suppose that Miles lists all possible (distinct) rearrangements of the letters in the word MATHEMATICS. He then picks one rearrangement at random. What is the probability that the first five letters of this rearrangement are ATTIC (in order)?

a) $\dfrac{1}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$
b) $\dfrac{2}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$
c) $\dfrac{4}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$
d) $\dfrac{6}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$
e) $\dfrac{8}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$

Solution: The eleven-letter word *MATHEMATICS* has three letters that are repeated. So the number of distinct rearrangements is $\dfrac{11!}{(2!)(2!)(2!)} = \dfrac{11!}{8}$. If *ATTIC* is at the beginning of one of the rearrangements there are now six letters left to complete the word with one letter appearing twice. So there are $\dfrac{6!}{2!} = \dfrac{6!}{2}$ distinct rearrangements that begin with *ATTIC*. The probability of choosing one of these arrangements at random is $\dfrac{6!}{2} \div \dfrac{11!}{8} = \dfrac{4 \cdot 6!}{11!} = \dfrac{4}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$.

The answer is **c**.

27. Suppose it costs \$ 1 to play the following game. You roll a die and flip a coin. If the coin lands on tails then you lose your dollar no matter what occurs with the die. If the coin lands on heads, and you roll an even number with the die, then you get your dollar back, but no more. If the coin lands on heads and you roll an odd number with the die, then you win your dollar back plus another \$5 dollars. On average, how much should you expect to win or lose (including the cost of playing) when playing this game?

a) Win \$0.75
b) Win \$0.50
c) Break even
d) Lose \$0.50
e) Lose \$0.75

Solution:

- 50% of the time the coin lands on tails and you lose the dollar.
- 25% of the time the coin lands on heads and the die is even in which case you don't lose or gain anything.
- 25% of the time the coin lands on heads and the die is odd in which case you gain \$5.

So on average you should expect to win $50\%(-\$1) + 25\%(\$0) + 25\%(\$5) = \$0.75$.

The answer is **a**.

28. Let S be the set of all positive integers n such that n^3 is a multiple of both 16 and 24. What is the largest integer that is a divisor of every integer n in S?

a) 6 b) 12 c) 18 d) 24 e) 216

Solution: If n^3 is divisible by 16, then the prime factorization of n must contain at least 2^2; otherwise $16 = 2^4$ will not divide n^3.

If n^3 is divisible by 24, then the prime factorization of n must contain at least 3^1 and 2^1.

The largest integer that is a divisor of all such n is $2^2 3^1 = 12$.

The answer is **b**.

29. Tyson is three times as old as Mandi. Two years ago, Tyson was four times as old as Mandi. How old is Mandi now?

a) 16 b) 14 c) 7 d) 6 e) 4

Solution: Let T equal Tyson's age now and M = Mandi's age now. Then

$$\begin{aligned} T &= 3M \\ T-2 &= 4(M-2) \end{aligned}$$

So $T = 3M = 4M - 6$, and $M = 6$.

The answer is **d**.

30. Suppose that

$$A \star B = 2A + 3B, \text{ and } C \heartsuit D = \frac{C^2 + 2D}{D}.$$

What is $(12 \star 4) \heartsuit 3$?

a) 432 b) 433 c) 434 d) 435 e) 436

Solution:

- $A \star B = 2A + 3B$, $12 \star 4 = 2 \cdot 12 + 3 \cdot 4 = 24 + 12 = 36$ and $(12 \star 4) \heartsuit 3 = 36 \heartsuit 3$
- $C \heartsuit D = \dfrac{C^2 + 2D}{D}$, so $36 \heartsuit 3 = \dfrac{36^2 + 2 \cdot 3}{3} = 12 \cdot 36 + 2 = 434$

The answer is **c**.

31. Solution 1 contains only liquids a and b in a ratio of $1:4$. Solution 2 contains also contains only liquids a and b, but in a ratio of $1:1$. Solution 3 is obtained by mixing Solutions 1 and 2 in a ratio of $5:1$. How many Tablespoons of liquid a are in 60 Tablespoons of Solution 3?

a) 15 b) 18 c) 20 d) 24 e) 30

Solution: Solution 1: The first two lines of the table show the number of units of liquid a and units of liquid b in 5 units of Solution 1 and 2 units of Solution 2. Doubling the total number of units of Solution 1 to 10, also doubles the amounts of liquid a and liquid b in 10 units of Solution 1. The amount of liquid a and liquid b in 12 units of Solution 3 can be found by adding rows 2 and 3 because the total amounts, 10 and 2, have a ratio of $5:1$. Thus the ratio of liquid a to liquid b in Solution 3 is $1:3$. So liquid a is 25% of Solution 3. Thus, 25% of 60 Tablespoons is 15 Tablespoons.

	units of liquid a	units of liquid b	total units
Solution 1	1	4	5
Solution 2	1	1	2
Solution 1	2	8	10
Solution 3	3	9	12

Solution 2: The first solution is $\frac{1}{5}$ liquid a and the second solution is $\frac{1}{2}$ liquid a. The third solution is $\frac{5}{6}$ solution 1 and $\frac{1}{6}$ solution 2. So the fraction of liquid a in solution 3 is $\frac{5}{6}\cdot\frac{1}{5}+\frac{1}{6}\cdot\frac{1}{2}=\frac{1}{4}$ and $\frac{1}{4}\cdot 60$ Tablespoons $=15$ Tablespoons.

The answer is **a**.

32. If the average test score for five students is 92, which of the following is the highest score a sixth student could get so that the average of all six scores would be no more than 86?

a) 55 b) 56 c) 57 d) 58 e) 59

Solution: If the average score for five students is 92, the sum of their scores is $5\cdot 92=460$. For the average score for six students to be 86, the sum of their scores must be $6\cdot 86=516$. If the sixth student got a score of $516-460=56$, then the average score would be exactly 86. If the sixth student got a score higher than 56, then the average for the six students would be more than 86. So the correct answer is 56.

The answer is **b**.

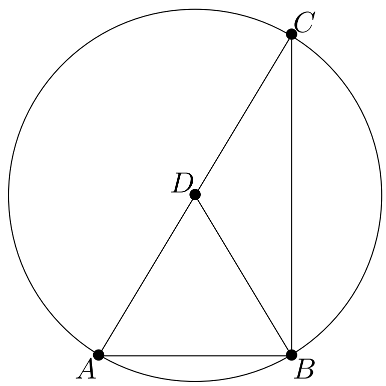
33. How many odd four-digit numbers are there that do not contain the digit 6?

a) 2560 b) 3240 c) 3645 d) 4050 e) 5000

Solution: For a four-digit number $abcd$ to be odd and not contain the digit 6, $a\in\{1,2,3,4,5,7,8,9\}$, $b,c\in\{0,1,2,3,4,5,7,8,9\}$ and $d\in\{1,3,5,7,9\}$. So there are 8 choices for a, 9 choices for each of b and c, and 5 choices for d. The total number of such four digit numbers is $8\cdot 9\cdot 9\cdot 5=3240$.

The answer is **b**.

34. Points A, B, and C lie on the circle. The point D is the center of a circle and lies on the line segment AC. If $AB = 6$ and $BD = 5$, find BC.

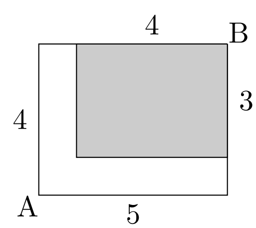


a) 6 b) 6.5 c) 7 d) 7.5 e) 8

Solution: $BD = 5$ is the radius of the circle, and AC is a diameter. So the diameter of the circle is 10. Triangle ABC is a right triangle with hypotenuse $AC = 10$ and leg $AB = 6$. By the Pythagorean Theorem, the remaining leg $BC = 8$.

The answer is **e**.

35. Two rectangles are given in the figure below so that two sides of the smaller rectangle are contained in two sides of the larger rectangle. The distances are given in miles. Anne and Bill both run from point A to point B each traveling at the same average speed without passing through the shaded smaller rectangle. Anne takes the lower route running straight to a corner of the shaded rectangle and then straight to B. Bill takes the upper route running straight to a corner of the shaded rectangle and then straight to B. If Anne takes 81 minutes, approximately how much longer will it take Bill to arrive?



a) 15 seconds b) 30 seconds c) 45 seconds

d) 1 minutes e) 4 minutes

Solution: Using the Pythagorean Theorem, Anne's distance in miles is $3 + \sqrt{26}$ and Bill's distance in miles is $4 + \sqrt{17}$. The ratio of Anne's time to Bill's time is $\dfrac{3+\sqrt{26}}{4+\sqrt{17}}$.

Let $f(x) = \sqrt{x}$. Then $f'(x) = \dfrac{1}{2\sqrt{x}}$. Using differentials, $\sqrt{26} - \sqrt{25} \approx f'(25) \cdot 1 = \dfrac{1}{10}$. So $\sqrt{26} \approx 5\dfrac{1}{10}$ and Anne's distance in miles is about $8\dfrac{1}{10}$. Using differentials, $\sqrt{17} - \sqrt{16} \approx f'(16) \cdot 1 = \dfrac{1}{8}$. So $\sqrt{17} \approx 4\dfrac{1}{8}$ and Bill's distance in miles is about $8\dfrac{1}{8}$. Thus, $\dfrac{3+\sqrt{26}}{4+\sqrt{17}} \approx \dfrac{8\frac{1}{10}}{8\frac{1}{8}} = \dfrac{\frac{81}{10}}{\frac{65}{8}} = \dfrac{81}{\frac{650}{8}} = \dfrac{81}{81\frac{1}{4}}$. So Bill's time is about $\dfrac{1}{4}$ minute or 15 seconds longer than Anne's. Notice that a calculator answer is 14.45 seconds which is very close to 15 seconds.

The answer is **a**.

36. Find the best estimate of the sum of the square roots of all the integers from 1 to 10,000, inclusive.

a) 7,071 b) 57,740 c) 577,394 d) 666,667 e) 707,107

Solution: The sum

$$\sum_{i=1}^{10{,}000} \sqrt{i}$$

can be approximated by left or right rectangular approximations.

So, it is approximately equal to the integral

$$\int_0^{10{,}000} \sqrt{x}dx = \frac{2}{3}x^{3/2}\Big|_0^{10{,}000} = \frac{2}{3} \cdot 1,000,000$$

The answer is **d**.

37. Let P be a third degree polynomial. If $P(1) = 1$, $P(2) = 1$, $P(3) = 7$, and $P(4) = 25$. Then $P(6) - P(5) =$

a) 30 b) 42 c) 56 d) 60 e) 90

Solution: Let $P_2(x) = P(x+1) - P(x)$, $P_1(x) = P_2(x+1) - P_2(x)$, and $P_0(x) = P_1(x+1) - P_1(x)$. Then $P_2(x), P_1(x)$ and $P_0(x)$ are polynomials of degree 2, 1, and 0, respectively. In particular, $P_0(x)$ is a constant. The solution to the problem is $P_2(5) = P(6) - P(5)$.

Consider the table:

$P(1)$		$P(2)$		$P(3)$		$P(4)$		$P(5)$		$P(6)$
	$P_2(1)$		$P_2(2)$		$P_2(3)$		$P_2(4)$		$P_2(5)$	
		$P_1(1)$		$P_1(2)$		$P_1(3)$		$P_1(4)$		
			$P_0(1)$		$P_0(2)$		$P_0(3)$			

We fill in the values that we know and that we can compute using the fact that each term is the difference of the two terms above it, and use the fact that $P_0(x)$ is constant.

1		1		7		25		$P(5)$		$P(6)$
	0		6		18		$P_2(4)$		$P_2(5)$	
		6		12		$P_1(3)$		$P_1(4)$		
			6		6		6			

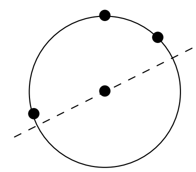
We now work our way back up using the fact that a term is the difference of the two terms above it.

1		1		7		25		$P(5)$		$P(6)$
	0		6		18		36		$P_2(5) = 60$	
		6		12		18		24		
			6		6		6			

The solution is $P_2(5) = 60$. It is straightforward but unnecessary to compute $P(5)$ and $P(6)$.

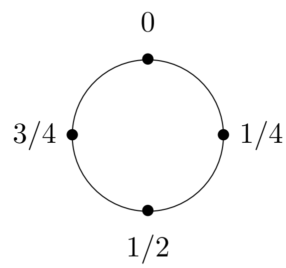
The answer is **d**.

38. If three points are scattered randomly on a circle, what is the probability that one can draw a line through the center of the circle, such that all three points lie on one side of the line?



a) $\frac{1}{2}$ b) $\frac{3}{4}$ c) $\frac{7}{8}$ d) $\frac{5}{8}$ e) $\frac{2}{3}$

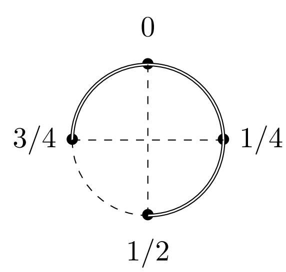
Solution: We can figure out this probability by labeling the three randomly chosen points A, B, and C. We can just rotate the circle so that point A is at the 'north pole' without changing the probability. Next, we worry about where point B lands. Let's label the circle so that if it lands at point A, we call that position 0. If it lands 90°clockwise on the circle from A, that is called 1/4, and so on.



We want to figure out, given the position of point B, what is the probability that point C will land in a position so that all three points will be on one side of some diameter of the circle.

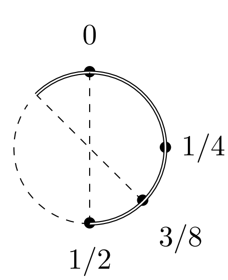
Notice that if B lands at 0, the three points will be on one side of a diameter no matter where C lands.

What if B lands at 1/4?



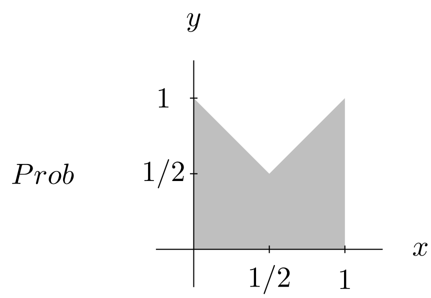
So, in this case, the probability that the three points land in a position so that all three points will be on one side of some diameter of the circle is 3/4.

What if B lands at 3/8?



So, in this case, the probability that the three points land in a position so that all three points will be on one side of some diameter of the circle is 5/8.

If we plot each of these probabilities, we get the following graph:

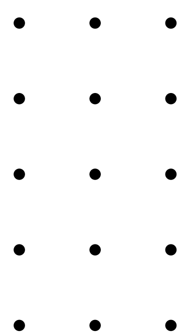


Position of B

The area of this region, which is also the probability we want, is 3/4.

The answer is **b**.

39. How many isosceles triangles can be drawn if each vertex must be one of the dots in the following square lattice?



a) at most 80

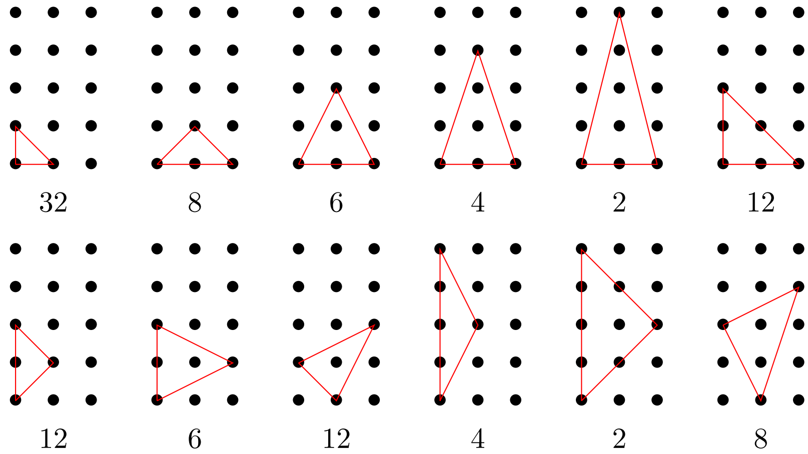
b) between 81 and 90, inclusive

c) between 91 and 100, inclusive

d) between 101 and 110, inclusive

e) greater than 110

Solution: Here are the various types of triangles, together with how often they occur:



There are 108, altogether.

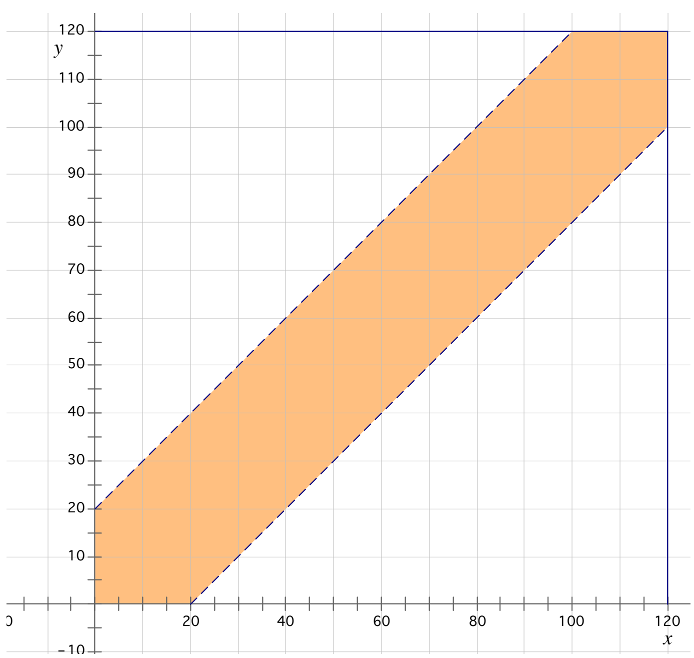
The answer is **d**.

40. Annabeth and Elinor are hoping to meet for dinner. They will each arrive at their favorite restaurant at a random time between 6:00 and 8:00 pm, stay for 20 minutes, and leave. What is the probability that they will see each other at the restaurant?

a) $\frac{7}{36}$ b) $\frac{9}{36}$ c) $\frac{11}{36}$ d) $\frac{13}{36}$ e) $\frac{15}{36}$

Solution: Measure time, t, in minutes starting at 6:00 pm. Let x be Annabeth's arrival time and y be Elinor's arrival time. Then $0 \leq x \leq 120$, $0 \leq y \leq 120$, and they meet when $x - 20 \leq y \leq x + 20$. The probability is the area of the shaded hexagonal region divided by the area of the 120×120 square.

$$\frac{120^2 - 100^2}{120^2} = \frac{12^2 - 10^2}{12^2} = \frac{44}{144} = \frac{11}{36}$$



The answer is **c**.

END OF EXAM