Utah State Mathematics Contest
Junior Exam
March 18, 2009

1. A metal cylinder is melted down and reshaped into a new cylinder. If the new diameter has been decreased by 20% without changing the volume, then the height has been increased by what percent?

   (a) 20 (b) 25 (c) 36 (d) 46.75 (e) 56.25

2. John has coins that may include pennies, nickels, dimes, or quarters. The mean value of the coins is 20 cents. If he were to add one quarter to his money, the new mean value would be 21 cents. How many quarters did he originally have?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

3. If the quadratic equation $x^2 - px - q = 0$ has two distinct real roots, then:

   (a) $p^2 - 4q \geq 0$ (b) $-p^2 - 4q > 0$ (c) $p^2 + 4q \geq 0$

   (d) $p^2 > 4q$ (e) $p^2 > -4q$

4. What is the fewest number of people you would need to have in a group if you needed to be certain that at least three of them have the same birth month?

   (a) 25 (b) 81 (c) 39 (d) 27 (e) 36

5. When Mary is 64 years old, her age is both a perfect square and a perfect cube. How many more years would she have to live until the next time her age would be both a perfect square and a perfect cube?

   (a) 61 (b) 448 (c) 152 (d) 665 (e) 17

6. In the November 2008 general election, Utah voters turned out in strong numbers. Between Utah and Salt Lake counties, 67% of the 840 thousand registered voters made it to the election booth. Of registered voters, 65% of those in Salt Lake County voted, while 72% of those in Utah County voted. How many thousands of registered voters were in Utah County?

   (a) 230 (b) 240 (c) 250 (d) 260 (e) 270

7. A digit is placed in each empty square in the grid so that each row and each column contains the digits 1, 2, 3, 4 and 5. What digit must be placed in the square marked 'X'?

	5	4		
1	3			
		5	3	
2		3	1	
				X

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

8. A man either was or will be N years old on his birthday in the year N^2. Assuming that the man was born in the 20^{th} century, what is the year that he was born?

(a) 1960 (b) 1980 (c) 1984 (d) 1992 (e) 2000

9. Ninety-six increased by 25% is the same as what number decreased by 25%?

(a) 160 (b) 120 (c) 192 (d) 144 (e) 200

10. Which of the following has the largest prime factor?

(a) 121 (b) 51 (c) 91 (d) 87 (e) 133

11. When Greg swims out from the beach, he is carried by the tide; it takes him 4 minutes to reach the nearest buoy. When he swims back in, he takes 16 minutes to swim against the tide. If Greg swims 100 yards per minute in still water, how many yards away is the buoy?

(a) 400 (b) 480 (c) 640 (d) 720 (e) 800

12. In an online game, you can choose from as many as four characters to be a part of your team. Assuming that, aside from yourself, you include at least one team member, how many different teams can you assemble from the available team members?

(a) 16 (b) 32 (c) 5 (d) 30 (e) 15

13. Which of the following plane figures is not necessarily convex?

(a) a regular hexagon (b) a parallelogram (c) a pentagon

(d) a scalene triangle (e) an ellipse

14. Ten acres of grass feeds twenty cows for sixty weeks. How many weeks would fifteen acres feed forty-five cows?

(a) 30 (b) 40 (c) 45 (d) 52.5 (e) 75

15. The point (6, 2) lies on a line with slope equal to $\frac{2}{5}$. What is the distance between (6, 2) and the closest point on the line with two integer coordinates?

(a) $\sqrt{29}$ (b) $\sqrt{53}$ (c) $2\sqrt{10}$ (d) $\sqrt{21}$ (e) $\sqrt{49}$

16. Determine the sum of all solutions to the following equation:

$$\sqrt{x + \sqrt{9 - x}} = 3$$

(a) 1 (b) 9 (c) 17 (d) 25 (e) 5

17. Solve the following inequality:

$$\frac{x-6}{x-3} \geq 2$$

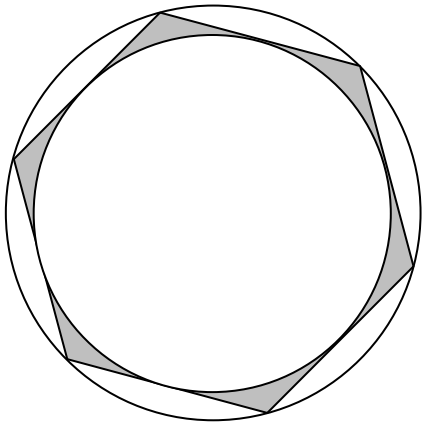
(a) $\{x \mid x \leq 0\}$ (b) $\{x \mid 6 \geq x > 3\}$ (c) $\{x \mid x \geq 6 \text{ or } 3 < x\}$

(d) $\{x \mid 3 > x \geq 0\}$ (e) $\{x \mid x > 3 \text{ or } 0 \leq x\}$

18. In the entire 7th grade class at Harbor Middle School, there are 85 students who take algebra and 92 who take American History; some of those mentioned are in both classes. There are 47 students who are enrolled in exactly one of the two. How many students are taking algebra, but not American History?

(a) 42 (b) 54 (c) 38 (d) 26 (e) 20

19. The picture to the right illustrates a circle inscribed in a regular hexagon inscribed in another circle. What is the ratio of the combined areas of the shaded regions to the combined areas of the unshaded regions?



(a) $\frac{6\sqrt{3} - 3\pi}{7\pi - 6\sqrt{3}}$ (b) $\frac{4\pi - 6\sqrt{3}}{12\sqrt{3} - \pi}$ (c) $\frac{6\sqrt{3} - 3\pi}{6\pi - 4\sqrt{3}}$

(d) $\frac{6\sqrt{3} - 3\pi}{4\pi - 6\sqrt{3}}$ (e) $\frac{1}{12}$

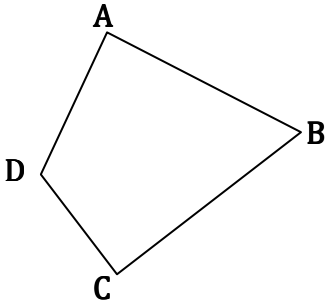
20. Jay is comparing fruit at a bazaar. He finds that his thirty-five dollars will evenly buy:
(1) nineteen pounds of bananas and five pounds of coconuts
(2) six pounds of bananas and twenty pounds of coconuts
How much more money would Jay need to buy fifteen pounds of each fruit?

(a) \$4.30 (b) \$2.50 (c) \$10.60 (d) \$7.00 (e) He has enough money

21. How many miles may a person ride on a bike, going at the rate of 8 miles per hour, if he is to be gone 11 hours, and if he must walk back at a rate of 3 miles per hour?

(a) 12 (b) 18 (c) 20 (d) 22 (e) 24

22. Given quadrilateral ABCD with $AB = 11$, $BC = 12$, $CD = 5$, $m\angle A = 90°$, and $m\angle C = 90°$, find the length of side AD.



(a) 6 (b) $4\sqrt{3}$ (c) $7\sqrt{2}$
(d) $3\sqrt{5}$ (e) $2\sqrt{10}$

23. Which of the following statements are true for all real values of x?
(I) $|x| \cdot |-x| = x^2$ (II) $|-x| = x$ (III) $|x^3| - |x^2| \geq 0$

(a) I (b) II (c) III (d) I & II (e) I, II & III

24. At 12:00 PM, the second, minute and hour hand on a clock all point to XII. Assuming continuous motion of all three hands, how many times will the second hand pass the minute hand in the time it takes for the minute hand to pass the hour hand three times?

(a) 192 (b) 193 (c) 194 (d) 195 (e) 196

25. Determine which of the following expressions is the least quantity:

(a) $2^{\left(2^{2^2}\right)}$ (b) $(2^2)^{\left(2^{2^2}\right)}$ (c) $\left(2^{2^2}\right)^{(2^2)}$ (d) $\left(2^{2^{2^2}}\right)^2$ (e) All are the same amount

26. A fox pursued by a hound has a head-start of 80 of her leaps. The fox makes five leaps while the hound makes only three; however, five hound leaps are equal to nine fox leaps. How many leaps will the hound take before he catches the fox?

(a) 1000 (b) 880 (c) 1200 (d) 1080 (e) 600

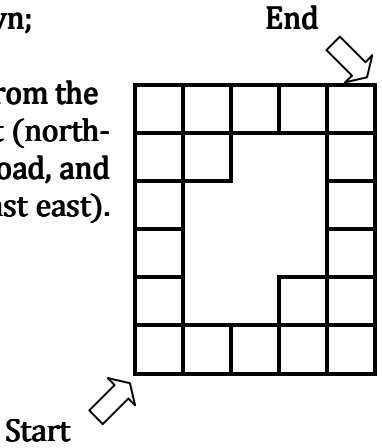
27. If there are two real numbers, X and Y, such that $X < -Y$, then $\sqrt{(X+Y)^2}$ is equivalent to:

(a) $X + Y$ (b) $-X - Y$ (c) $X - Y$ (d) $Y - X$ (e) none of these

28. Sarah has achieved times of 86, 94, 97, 88, and 96 for swimming 100 meters on her last five attempts. How many seconds would she need on her sixth try to bring her median time down to 92 seconds?

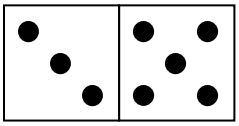
(a) 92 (b) 87 (c) 91 (d) 90 (e) 89

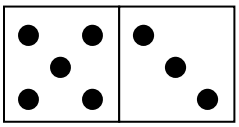
29. The grid to the right represents a road map of a very small town; the black lines are roads while the spaces represent city blocks, with a large park in the middle. You wish to travel from the bottom left (south-west) corner of the map to the top right (north-east) corner. Assume that you must travel exclusively by road, and that you must always travel up or to the right (north or east east). By how many different paths can you accomplish this goal?



(a) 30 (b) 156 (c) 102 (d) 120 (e) 84

30. The game of dominoes is made up of rectangular game pieces, each of which has on its upper surface two squares, each containing from 0 to 12 dots. We can represent the domino with three dots in one square and five dots in the other as both the ordered pair (5, 3) and (3, 5) if it is rotated 180° as shown. These two would be considered to be the same domino. Find the maximum number of distinct dominoes which can be created.





(a) 169 (b) 98 (c) 144 (d) 78 (e) 91

31. The product of $\sqrt[4]{25}$ and $\sqrt[5]{125}$ is equal to:

(a) $25\sqrt[9]{5}$ (b) $5\sqrt[9]{5}$ (c) $5\sqrt[20]{125}$ (d) $5\sqrt[10]{5}$ (e) $25\sqrt[10]{5}$

32. In decimal notation (base 10), the number two-thousand and nine has how many digits in binary notation (base 2)?

(a) 10 (b) 11 (c) 12 (d) 13 (e) More than 13

33. Assume that you have 12 poles, each measuring a different whole number of meters in length and ranging from 1 meter to 12 meters in length. How many distinct combinations of 3 poles can join end to end to form a triangle?

(a) 95 (b) 72 (c) 87 (d) 154 (e) 100

34. Determine which factor evenly divides the following polynomial expression:

$$x^3 - 2x^2 - 9x + 60$$

(a) $x + 2$ (b) $x - 3$ (c) $x + 4$ (d) $x - 5$ (e) $x + 6$

35. You have two marble bags – one leather and the other cotton. The leather bag contains three red and three black marbles. The cotton bag contains six red marbles. You randomly select one marble from each bag and exchange them. If you were to now select a new marble from the leather bag, the probability that it would be red is:

(a) $\frac{11}{18}$ (b) $\frac{2}{3}$ (c) $\frac{19}{36}$ (d) $\frac{1}{2}$ (e) $\frac{7}{12}$

36. The following "proof" showing that $+1 = -1$ has a mis-statement in which step?

$$+1 \overset{(1)}{=} \sqrt{1^2} \overset{(2)}{=} \sqrt{(-1)^2} \overset{(3)}{=} \sqrt{(-1)\cdot(-1)} \overset{(4)}{=} \sqrt{(-1)}\cdot\sqrt{(-1)} \overset{(5)}{=} i\cdot i \overset{(6)}{=} i^2 \overset{(7)}{=} -1$$

(a) 2 (b) 3 (c) 4 (d) 5 (e) 7

37. If $2^x = 15$, and if $15^y = 32$, then what is the value of xy?

(a) $\frac{9}{2}$ (b) 5 (c) $\frac{16}{3}$ (d) 6 (e) $\frac{13}{2}$

38. Find the area of the smallest region bounded by the graphs of the equations:

$y = |x|$ and $x^2 + y^2 = 20$

(a) 5π (b) 25π (c) 20π (d) 4π (e) $\frac{25}{16}\pi$

39. Simplify the expression:

$$\frac{35! + 34!}{35! - 34!}$$

(a) $\frac{18}{17}$ (b) 2346 (c) $\frac{15}{13}$ (d) 69 (e) $\frac{7}{5}$

40. Five women are playing a card game in which they each have five cards. Each card is one of five different colors. All of the following statements are true:
(I) Any player who has a yellow card also has an orange card.
(II) Only if a player has a yellow and a red card does she have a blue card.
(III) A player has a green card if she does not have a yellow card.
(IV) A player does not have a blue card only if she does not have an orange card.
(V) Of blue, green, yellow and orange cards, each player has at least two colors.
Only one player currently has one card of each color. This woman is the only player holding a card that is:

(a) Red (b) Orange (c) Yellow (d) Green (e) Blue

2009 Utah State Mathematics Contest
Junior Exam Solutions

1. (e) 56.25

$V = \pi r^2 h$. If the diameter is down 20%, the radius is down 20%. To maintain the same volume, the equation describing the adjusted lengths would be $V = \pi(\frac{4}{5} r)^2 (\frac{25}{16} h)$. Since $\frac{25}{16} = 1.5625$, there must be an increase of 56.25%.

2. (d) 3

Let X be the initial number of coins. Since their average value is 20 cents, their total value is 20X. By adding a quarter, we get $\frac{20X + 25}{X + 1} = 21$. Solving for X, there were originally 4 coins, totaling 80 cents in value. They must have been three quarters and a nickel.

3. (e) $p^2 > -4q$

The discriminant of the quadratic formula is $(b^2 - 4ac)$, which in this case is $(p^2 + 4q)$. Since the discriminant must be positive for a quadratic equation to have two distinct real roots, $p^2 + 4q > 0$, or $p^2 > -4q$.

4. (a) 25

You would need to have two people born in each of the twelve months, plus one additional person, to guarantee that three people were born in the same month.

5. (d) 665

For an integer to be both a square and a cube, it must be a perfect sixth power. $2^6 = 64$, and $3^6 = 729$. $729 - 64 = 665$ more years until her age is a square and a cube again.

6. (b) 240

Let X be the number of voters from Salt Lake County, and Y be the number of voters from Utah County, both measured in thousands. $X + Y = 840$, and $0.65X + 0.72Y = 0.67(840)$. Solving this system of equations yields that $X = 600$, $Y = 240$.

7. (c)

3	5	4	2	1
1	3	2	5	4
4	1	5	3	2
2	4	3	1	5
5	2	1	4	3

8. (b) 1980

If the man is alive in a year which is a square, we need a square between 1900 and 2100. The only available candidates are $44^2 = 1936$ and $45^2 = 2025$. If a man is 44 in 1936, he was born in 1892, which is in the 19th century. If a man was is 45 in 2025, he was born in 1980.

9. (a) 160

Increasing 96 by 25% is analogous to multiplying by $\frac{5}{4}$. $96 \cdot \frac{5}{4} = 120$. Decreasing a number by 25% is analogous to multiplying by $\frac{3}{4}$; reversing this process involves multiplying by $\frac{4}{3}$. $120 \cdot \frac{4}{3} = 160$.

10. (d) 87

$121 = 11 \cdot 11$; $51 = 3 \cdot 17$; $91 = 7 \cdot 13$; $87 = 3 \cdot 29$; $133 = 7 \cdot 19$

11. (c) 640

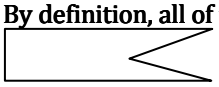
Tide speed = X. Speed with tide = (100 + X). Speed against tide = (100 – X).
$4(100 + X) = 16(100 - X)$. Solving for X gives that $X = 60$. Swimming 4 minutes at 160 yards per minute will move Greg 640 yards.

12. (e) 15

Of the four available additional team members, choosing each member is a yes/no choice, giving $2^4 = 16$ different ways to select from them. However, since you must eliminate the case where you choose nobody (you must choose at least one), there are only 15 choices.

13. (c) a pentagon

By definition, all of the other shapes are convex. The following is a concave pentagon.



14. (b) 40

Cows and acres vary directly. Cows and weeks vary inversely. Acres and weeks vary directly. Let C be Cows, A be Acres, W be weeks, and K be an unknown constant. The consumption is guided by the following equation: $KA = CW$. Using the given data, K must be 120. If $A = 15$ and $C = 45$, then $W = 40$.

15. (a) $\sqrt{29}$

Using the slope, the two nearest points to (6, 2) on this line which have integer coordinates are (1, 0) and (11, 4). Using the distance formula, the distance to either point is $\sqrt{(5)^2 + (2)^2} = \sqrt{29}$.

16. (c) 17

Squaring both sides and subtracting X, $\sqrt{9 - X} = 9 - X$. Again squaring and maneuvering into standard form, $X^2 - 17X + 72 = 0$. X is either 8 or 9. Both solutions check out, and sum to 17.

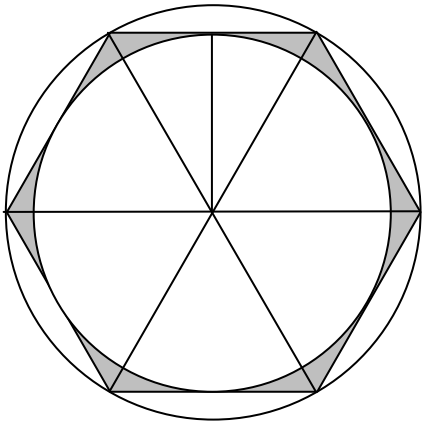
17. (d) $\{x \mid 3 > x \geq 0\}$

Convert into standard form and induce a common denominator to produce $\frac{x-6}{x-3} - \frac{2x-6}{x-3} \geq 0$; combining fractions, $\frac{-x}{x-3} \geq 0$, or $\frac{x}{x-3} \leq 0$. For a fraction to be less than zero, the numerator and denominator must have opposite signs. Between 0 and 3, x is positive while $(x - 3)$ is negative. However, x cannot be 3, as this would render the fraction as undefined.

18. (e) 20

Let A = # of students in only History, B = # of students in only Algebra, C = # of students in both. $(A + C) = 92$, $(B + C) = 85$, $(A + B) = 47$. Solving this system of equations gives that $C = 65$, $A = 27$, and $B = 20$.

19. (a) $\frac{6\sqrt{3} - 3\pi}{7\pi - 6\sqrt{3}}$



A regular hexagon can be divided into six equilateral triangles. Each side length is equal to the larger radius, R. The height of each equilateral triangle is the smaller radius, equal to $\frac{\sqrt{3}}{2}R$. The shaded area is equal to the difference between the area of the hexagon and the area of the smaller circle $= 6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} R^2 - \frac{3\pi}{4}R^2$. The unshaded area is equal to the area of the large circle minus the shaded area $= \pi R^2 - (6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} R^2 - \frac{3\pi}{4}R^2)$. The ratio of the two areas is, when simplified, equal to $\frac{6\sqrt{3} - 3\pi}{7\pi - 6\sqrt{3}}$.

20. (d) \$7.00

Let X be the price per pound of bananas, and let Y be the price per pound of coconuts. The given information leads to $19X + 5Y = 35$, $6X + 20Y = 35$. Solving for X and Y, $X = 1.50$ and $Y = 1.30$. Then, $15(X + Y) = 15(2.80) = \$42.00$. He needs \$7 more.

21. (e) 24

Let X be the distance travelled out. For uniform motion, Time = Speed/Distance, so $\frac{X}{8}$ is the time spent outbound, and $\frac{X}{3}$ is the time heading back. $\frac{X}{8} + \frac{X}{3} = 11$. Solving for X, the distance must be 24 miles.

22. (b) $4\sqrt{3}$

By the Pythagorean Theorem, BD = 13. Also by the Pythag. Thm., AD = $\sqrt{13^2 - 11^2} = \sqrt{48} = 4\sqrt{3}$

23. (a) I

$|x| \cdot |-x| = (x)\cdot(x)$ or $(-x)\cdot(-x)$, both of which = x^2.
$|-x| = (x)$ or $(-x)$, depending on the sign of x
$|x^3| - |x^2| \geq 0$, only if $|x| \geq 1$, or if $x = 0$.

24. (b) 193

Let X be the amount of time after 3 PM when the minute hand crosses the hour hand, measured in minutes. The equation to find X is: $X = 15 + \frac{X}{60} \cdot 5$. Solving, $X = \frac{180}{11}$, or $16\frac{4}{11}$, or about 16 minutes, 22 seconds. The second hand overlaps the minute hand 59 times per hour. From noon to 3 o'clock, that makes 177 overlaps. From 3PM to 3:16 PM, the second hand will overlap the minute hand 15 more times. At 3:16:22, the second hand will have passed over the minute hand one additional time (at about 3:16:16).

25. (c)

$2^{\left(2^{2^{2^2}}\right)} = 2^{2^{16}}$; $(2^2)^{(2^{2^2})} = 2^{2^5}$; (c) $\left(2^{2^2}\right)^{(2^2)} = 2^{2^4}$; (d) $\left(2^{2^{2^2}}\right)^2 = 2^{2^5}$

26. (e) 600

The fox makes 25 fox leaps in the time that the hound makes 15 hounds leaps. However, 15 hound leaps are equal in length to 27 fox leaps. So, every time the fox makes 25 fox leaps, the hound gains 2 fox leaps on the fox. Since $\frac{80}{2} = 40$, the fox will take 25 fox leaps 40 times, or make 1000 fox leaps. This will take as long as 600 hound leaps, which are equal in length to 1080 fox leaps.

27. (b) –X – Y

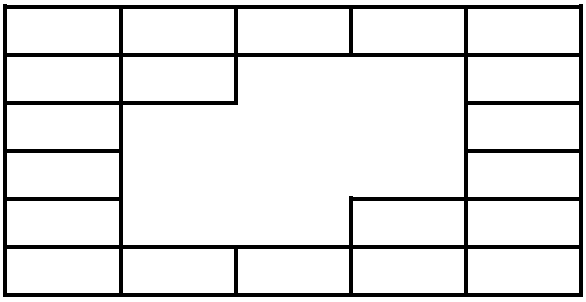
If $X < -Y$, then $X + Y < 0$. The square root of the square of a negative quantity is the opposite of the original quantity.

28. (d) 90

In numerical order, the current times are 86, 88, 94, 96, 97, and the current median is 94. If the median, after a sixth time, is to be 92, then the new value, X must be part of calculating the median. The new set of data would be 86, 88, X, 94, 96, 97. $\frac{X+94}{2} = 92$; solving for X leads to $X = 90$.

29. (c) 102

The number of paths to a specific corner is equal to the sum of the ways to arrive at the two corners which feed into that corner. The corners on the far left and bottom row each have only one path. The remaining corners have path totals working very much like Pascal's triangle.



30. (e) 91

The number of dominoes that have different numbers on the two squares would be $\frac{13 \cdot 12}{2 \cdot 1} = 78$. Then, there are thirteen more dominoes that have twin squares (squares with the same number of dominoes on each).

31. (d) $5\sqrt[10]{5}$

Using rational exponent notation and a common base of 5, you have $5^{\frac{2}{4}} \cdot 5^{\frac{3}{5}} = 5^{\frac{11}{10}} = 5\sqrt[10]{5}$

32. (b) 11

Converting 2009 to binary (base 2) notation would require knowing all of the powers of 2 less than or equal to 2009. However, for a digit count, it is only necessary to know the largest power of 2 less than 2009; $2^{10} = 1024$, while $2^{11} = 2048$. Thus, to write 2009 in binary, you would need place values from 2^0 to 2^{10}, or 11 place values/digits.

33. (a) 95

First, the foundation of the problem is that the two shortest legs of a triangle must have a sum greater than the longest leg. The pole of length 1 cannot be used at all for this reason. Let X be the length of the smallest pole in your triangle, and Y be the difference between the lengths of the middle and longest poles. For example, the triangle made with pole lengths 4, 8, 10 would have (X, Y) = (4, 2). X must be no greater than 10. The chart values indicate the number of triangles which can be created for a given (X, Y). The sum of all possible triangles is 95.

	X = 2	X = 3	X = 4	X = 5	X = 6	X = 7	X = 8	X = 9	X = 10
Y = 1	9	8	7	6	5	4	3	2	1
Y = 2		7	6	5	4	3	2	1	
Y = 3			5	4	3	2	1		
Y = 4				3	2	1			
Y = 5					1				

34. (c) x + 4

The Remainder Theorem states that if (x – c) evenly divides f(x), then f(c) = 0. While any of the five options are possible trial divisors, only $(-4)^3 - 2(-4)^2 - 9(-4) + 60 = 0$.

35. (e) $\frac{7}{12}$

The leather bad had probability of $\frac{3}{6}$ of trading a black marble for a red, and probability of $\frac{3}{6}$ of maintaining the same red-black mix. The probability of drawing a red now is $\frac{4}{6} \cdot \frac{3}{6} + \frac{3}{6} \cdot \frac{3}{6} = \frac{21}{36} = \frac{7}{12}$.

36. (c) 4

$\sqrt{(-1) \cdot (-1)} \neq \sqrt{(-1)} \cdot \sqrt{(-1)}$. The Product Property of Radicals ($\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$) does not apply if ($\sqrt[n]{a}$) and ($\sqrt[n]{b}$) are not real numbers.

37. (b) 5

$2^{xy} = (2^x)^y = 15^y = 32$, , then (xy) must be equal to 5.

38. (a) 5π

The region bounded by the two equations is one quarter of a circle centered at the origin with radius equal to $\sqrt{20}$. The area of such a figure is $\frac{1}{4} \cdot 20\pi = 5\pi$.

39. (a) $\frac{18}{17}$

By factoring out the GCF in the numerator and denominator, you get $\frac{34!(35+1)}{34!(35-1)} = \frac{36}{34} = \frac{18}{17}$

40. (d) Green

(I) implies that if a player has a yellow card, then she also has an orange card. (II) implies that if a player has a blue card, then she also has a yellow and a red card. (IV) implies that if a player has an orange card, then she also has a blue card. (V) implies that each player must have at least one of orange, yellow, and blue. Since yellow implies orange, orange implies blue, and blue implies yellow, then every player must have yellow, orange and blue. Since each player has blue, each player also has red. Thus, all players have red, orange, yellow and blue. If one player is the only player holding all five colors, then that player uniquely holds green, as everybody else has all other colors. (III) implies only that each player has yellow and/or green.