Statistics Prelim Exam
University of Utah
Department of Mathematics

May 2020

Read the following instructions before you begin:

- You may attempt all of 10 problems in this exam. As part of your submission include a list of the problems you are turning in and want graded (writing this on a separate piece of paper and scanning and submitting it with the rest of the exam is sufficient).
- Each problem is worth 10 points; 60 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 5080–5090/6010 texts, then you need to carefully state that result.

Exam problems begin here:

1. Let $U_1, U_2, \ldots, U_n$ be independent and identically distributed random variables, uniform on $[0,1]$. Let $U_{1,n} \le U_{2,n} \le \ldots \le U_{n,n}$ be the order statistics. Show that $nU_{3,n}$ converges in distribution and compute the limit.

2. Let X, Y be independent random variables with respective densities

$$f(t) = \begin{cases} 0, & \text{if } t \notin [-2,3] \\ \frac{1}{5}, & \text{if } t \in [-2,3] \end{cases}$$

and

$$g(t) = \begin{cases} 0, & \text{if } t \notin [-1,5] \\ \frac{1}{6}, & \text{if } t \in [-1,5] \end{cases}$$

Compute the density of $X+Y$.

3. Let X and Y be independent normally distributed random variables with $EX = \mu_1, \operatorname{Var} X = \sigma_1^2, EY = \mu_2$ and $\operatorname{Var} Y = \sigma_2^2$. Show that $X+Y$ and $X-Y$ are independent if and only if $\sigma_1^2 = \sigma_2^2$.

4. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential(λ) random variables, meaning that their pdf is

$$f(t) = \begin{cases} 0, & \text{if } -\infty < t < 0 \\ \frac{1}{\lambda} e^{-t/\lambda}, & \text{if } 0 \le t < \infty. \end{cases}$$

Find the uniformly minimum variance unbiased estimator for $\tau = \lambda^5$.

5. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with density function

$$f(t) = \begin{cases} 0, & \text{if } -\infty < t < \theta \\ \dfrac{2\theta}{(\theta+t)^2}, & \text{if } \theta \le t < \infty, \end{cases}$$

for $\theta > 0$ a parameter. Find the maximum likelihood estimator for θ.

6. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with density function

$$h(t;\theta) = \frac{3t^2}{\theta^3} \mathbf{1}\left\{0 \le t \le \theta\right\},$$

where $\theta > 0$ is a parameter. Find a $100(1-\alpha)\%$, two-sided, equal-tailed confidence interval for θ.

7. Let $X_1, \ldots, X_n$ be independent and identically distributed $N(\mu, \sigma^2)$, where μ, σ^2 are both unknown. We wish to test $H_0 : \sigma > \sigma_0$ against $H_a : \sigma \le \sigma_0$. We reject H_0 if $S^2 < c$, where

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2, \quad \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Find c such that the size of the test is α.

8. Let $X_1, \ldots, X_n$ be independent and identically distributed with density function

$$h(t;\theta) = \theta t^{\theta-1} \mathbf{1}\left\{0 \le t \le 1\right\}$$

for $\theta > 0$ unknown. Use the Neyman-Pearson Lemma to find the uniformly most powerful test of size α for $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_1$, where $\theta_1 > \theta_0$.

9. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables. The density function of X_i is

$$g(t;\theta_i) = \theta_i t^{-\theta_i - 1} \mathbf{1}\left\{t \ge 1\right\}.$$

Here $\theta_i > 0$ is an unknown parameter. We wish to test

$$H_0 : \theta_1 = \theta_2 = \ldots = \theta_n$$

against the alternative that H_0 is not true. Find a test using the generalized likelihood ratio.

10. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with (unknown) density function F. Let

$$F_0(t) = (1 - e^{-t}) \mathbf{1}\left\{t > 0\right\}.$$

We wish to test $F(t) = F_0(t)$ for all t. Explain the application of the χ^2 goodness-of-fit test.

Special Continuous Distributions

Notation and Parameters	Continuous pdf $f(x)$	Mean	Variance	MGF $M_X(t)$
Student's t				
$X \sim t(\nu)$	$\dfrac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \dfrac{1}{\sqrt{\nu\pi}} \left(1+\dfrac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$	0	$\dfrac{\nu}{\nu-2}$	**
$\nu = 1, 2, \ldots$		$1 < \nu$	$2 < \nu$	
Snedecor's F				
$X \sim F(\nu_1, \nu_2)$	$\dfrac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\dfrac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} x^{\frac{\nu_1}{2}-1}$	$\dfrac{\nu_2}{\nu_2-2}$	$\dfrac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$	**
$\nu_1 = 1, 2, \ldots$ $\nu_2 = 1, 2, \ldots$	$\times \left(1+\dfrac{\nu_1}{\nu_2}x\right)^{-\frac{\nu_1+\nu_2}{2}}$	$2 < \nu_2$	$4 < \nu_2$	
Beta				
$X \sim \text{BETA}(a, b)$	$\dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$	$\dfrac{a}{a+b}$	$\dfrac{ab}{(a+b+1)(a+b)^2}$	*
$0 < a$ $0 < b$	$0 < x < 1$			

*Not tractable.
**Does not exist.

Special Continuous Distributions

Notation and Parameters	Continuous pdf $f(x)$	Mean	Variance	MGF $M_X(t)$
Weibull				
$X \sim \text{WEI}(\theta, \beta)$	$\frac{\beta}{\theta^\beta} x^{\beta-1} e^{-(x/\theta)^\beta}$	$\theta\Gamma\left(1+\frac{1}{\beta}\right)$	$\theta^2\left[\Gamma\left(1+\frac{2}{\beta}\right)-\Gamma^2\left(1+\frac{1}{\beta}\right)\right]$	*
$0<\theta$ $0<\beta$	$0<x$			
Extreme Value				
$X \sim \text{EV}(\theta, \eta)$	$\frac{1}{\theta}\exp\{[(x-\eta)/\theta]-\exp[(x-\eta)/\theta]\}$	$\eta-\gamma\theta$	$\frac{\pi^2\theta^2}{6}$	$e^{\eta t}\Gamma(1+\theta t)$
$0<\theta$		$\gamma \doteq 0.5772$ (Euler's const.)		
Cauchy				
$X \sim \text{CAU}(\theta, \eta)$	$\frac{1}{\theta\pi\{1+[(x-\eta)/\theta]^2\}}$	**	**	**
$0<\theta$				
Pareto				
$X \sim \text{PAR}(\theta, \kappa)$	$\frac{\kappa}{\theta(1+x/\theta)^{\kappa+1}}$	$\frac{\theta}{\kappa-1}$	$\frac{\theta^2\kappa}{(\kappa-2)(\kappa-1)^2}$	**
$0<\theta$ $0<\kappa$	$0<x$	$1<\kappa$	$2<\kappa$	
Chi-Square				
$X \sim \chi^2(\nu)$	$\frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}$	ν	2ν	$\left(\frac{1}{1-2t}\right)^{\nu/2}$
$\nu = 1, 2, \ldots$	$0<x$			

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Notation and Parameters	Discrete pdf $f(x)$	Mean	Variance	MGF $M_X(t)$
Binomial				
$X \sim \text{BIN}(n, p)$	$\binom{n}{x}p^x q^{n-x}$	np	npq	$(pe^t + q)^n$
$0 < p < 1$ $q = 1 - p$	$x = 0, 1, \ldots, n$			
Bernoulli				
$X \sim \text{BIN}(1, p)$	$p^x q^{1-x}$	p	pq	$pe^t + q$
$0 < p < 1$ $q = 1 - p$	$x = 0, 1$			
Negative Binomial				
$X \sim \text{NB}(r, p)$	$\binom{x-1}{r-1}p^r q^{x-r}$	r/p	rq/p^2	$\left(\frac{pe^t}{1 - qe^t}\right)^r$
$0 < p < 1$ $r = 1, 2, \ldots$	$x = r, r+1, \ldots$			
Geometric				
$X \sim \text{GEO}(p)$	pq^{x-1}	$1/p$	q/p^2	$\frac{pe^t}{1 - qe^t}$
$0 < p < 1$ $q = 1 - p$	$x = 1, 2, \ldots$			
Hypergeometric				
$X \sim \text{HYP}(n, M, N)$	$\binom{M}{x}\binom{N-M}{n-x}\Big/\binom{N}{n}$	nM/N	$n\frac{M}{N}\left(1 - \frac{M}{N}\right)\frac{N-n}{N-1}$	*
$n = 1, 2, \ldots, N$ $M = 0, 1, \ldots, N$	$x = 0, 1, \ldots, n$			
Poisson				
$X \sim \text{POI}(\mu)$	$\frac{e^{-\mu}\mu^x}{x!}$	μ	μ	$e^{\mu(e^t - 1)}$
$0 < \mu$	$x = 0, 1, \ldots$			
Discrete Uniform				
$X \sim \text{DU}(N)$	$1/N$	$\frac{N+1}{2}$	$\frac{N^2 - 1}{12}$	$\frac{1}{N}\frac{e^t - e^{(N+1)t}}{1 - e^t}$
$N = 1, 2, \ldots$	$x = 1, 2, \ldots, N$			

*Not tractable.