Department of Mathematics
University of Utah
Real and Complex Analysis Preliminary Examination
May 2020

Instructions. Attempt as many problems as you want. To pass, you must demonstrate mastery of both real and complex analysis. Getting 3 problems completely correct on each section is sufficient to do this. Carefully state all theorems you are using.

Part A: Real Analysis

Problem 1. Let $(X, \mathcal{M})$ be a measurable space and μ, ν two measures on it. Suppose that for every $E \in \mathcal{M}$

$$\mu(E) = 0 \quad \Rightarrow \quad \nu(E) = 0$$

(a) Assuming $\nu(X) < \infty$ prove that for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\mu(E) < \delta \quad \Rightarrow \quad \nu(E) < \epsilon$$

You are not allowed to use the Radon-Nikodym theorem for this problem.

(b) Find an example where the above statement is false when $\nu(X) = \infty$.

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to [0, \infty)$ be a measurable function. Define a new measure ν by

$$\nu(E) = \int_E f \, d\mu$$

for $E \in \mathcal{M}$ (you don't have to prove that ν is a measure). Prove that for every positive measurable function g

$$\int_X g \, d\nu = \int_X fg \, d\mu$$

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Denote by S the set of equivalence classes of measurable functions $X \to \mathbb{R}$ where $f \sim g$ if and only if $f = g$ a.e. For $f, g \in S$ define

$$d(f, g) = \int_X \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx$$

Show that (S, d) is a metric space and $f_k \to f$ in (S, d) if and only if $f_k \to f$ in measure in X.

Problem 4. Let $C([0, 1])$ be the Banach space of continuous functions on $[0, 1]$ with the sup norm and view it as a natural subspace of $L^\infty([0, 1])$, where $[0, 1]$ is given Lebesgue measure, and the norm is essential supremum.

Prove the existence of a bounded functional on $L^\infty([0, 1])$ which is not identically 0 but vanishes on $C([0, 1])$.

Problem 5. Let $f_n \in C([0,1])$ for $n = 1, 2, \cdots$. Show that the following statements are equivalent:

(a) For every $\lambda \in C([0,1])^*$ we have $\lambda(f_n) \to 0$ as $n \to \infty$.

(b) $f_n(x) \to 0$ for every $x \in [0,1]$ and $\sup ||f_n||_\infty < \infty$.

Here $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$.

Problem 6. Let $\mathcal{H}$ be a Hilbert space and $P : \mathcal{H} \to \mathcal{H}$ a self-adjoint operator, i.e. $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in \mathcal{H}$. Also assume that $P^2 = P$. Prove that P is orthogonal projection to a closed subspace. Note that we are not assuming that P is bounded.

Part B: Complex Analysis

Problem 7. Let $f(z)$ be an analytic function. Show that the successive derivatives of $f(z)$ at a point can never satisfy $|f^{(n)}(z)| > n!n^n$ for all n.

Problem 8. Evaluate the integral by the method of residue:

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}dx.$$

Problem 9. If $f(z)$ is analytic in $|z| \leq 1$ and satisfies $|f| = 1$ on $|z| = 1$, show that $f(z)$ is rational.

Problem 10. Show that the family of functions $\{z^n\}_{n\in\mathbb{Z}_{\geq 0}}$ form a normal family in $|z| < 1$, also in $|z| > 1$, but not in any region that contains a point on the unit circle.

Problem 11. Let $\wp(z)$ be the Weierstrass $\wp$ function. Prove that there are constants g_2 and g_3 such that

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$