PhD Preliminary Qualifying Examination: Applied Mathematics II (6720)

May 2022

Instructions: Answer three out of five questions. Indicate clearly which questions you wish to be graded.

1. Let C be the unit circle centered at the origin. Cauchy's integral formula implies that

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} d\xi, \quad |z| < 1.$$

   (a) Perform the change of variables $\xi = \mathrm{e}^{i\theta}$ to show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z} d\theta,$$

   where z lies inside the unit circle. Explain why

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - 1/\bar{z}} d\theta.$$

   (b) Using the fact that $\xi = 1/\bar{\xi}$ on the unit circle, show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left(\frac{\xi}{\xi - z} \pm \frac{\bar{z}}{\bar{\xi} - \bar{z}} \right) d\theta.$$

   (c) Taking the plus sign in part (b), deduce the Poisson formula for the real part $u(r, \phi)$ of $f(z)$ with $z = r\mathrm{e}^{i\phi}$:

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2} u(1, \theta) d\theta.$$

   This shows that the real part of $f(z)$ on the unit circle determines the real part of $f(z)$ everywhere inside the unit circle.

2. Use a sector contour with radius R centered at the origin with angle $0 \le \theta \le 2\pi/5$ to find, for $a > 0$,

$$\int_0^{\infty} \frac{dx}{x^5 + a^5} = \frac{\pi}{5a^4 \sin(\pi/5)}$$

3. (a) Use principal value integrals to show that

$$\int_0^{\infty} \frac{\cos kx - \cos mx}{x^2} dx = -\frac{\pi}{2}(k - m), \ k, m > 0.$$

   (b) Deduce from (a) that

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

4. Determine the conformal map f that takes the region D exterior to two circular cylinders of radius R to the region outside a circular disc of unit radius centered at the origin (see figure 1). Use the following steps:

   (a) Show how the map $\omega = i - 2R/z$ takes the region D to the infinite strip $0 < \mathrm{I}m\omega < 2i$.

   (b) Show how the map $\xi = \mathrm{e}^{\pi\omega/2}$ takes the infinite strip to the upper-half complex ξ-plane.

   (c) Show how the map $\eta = (\xi + i)/(\xi - i)$ takes the upper-half complex ξ-plane to the region outside the unit disc, that is, $|\eta| > 1$.

   (d) Find f.

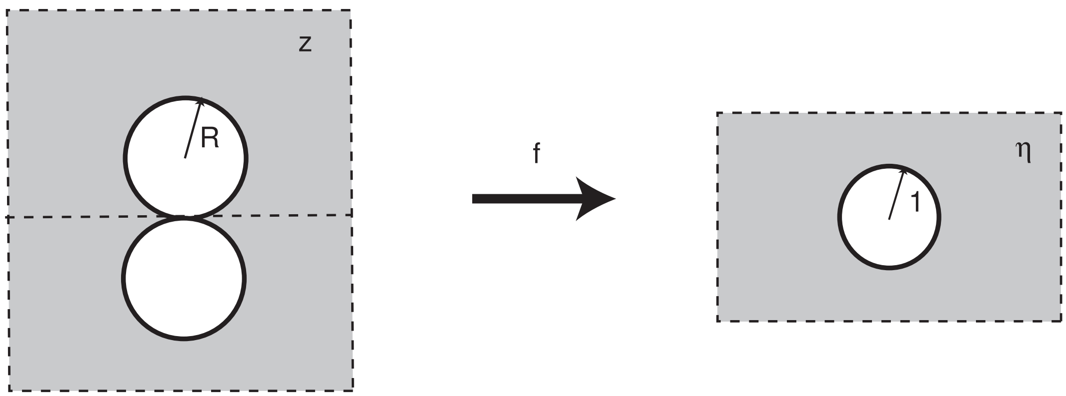

Figure 1: Find the conformal map f

5. (a) Discuss the flow pattern around a circular obstacle associated with the complex potential

$$\Omega(z) = u_0\left(z + \frac{a^2}{z}\right) + \frac{i\gamma}{2\pi}\log z.$$

   (b) Show that $r = a$ is a streamline.

   (c) Determine the asymptotic velocity when $z \to \infty$. Sketch the flow when $\gamma = 0$.

   (d) Use the Blasius formula

$$F_x - iF_y = \frac{1}{2}i\rho\int_C\left(\frac{d\Omega}{dz}\right)^2 dz$$

   to determine the lift on the obstacle when $\gamma \neq 0$.