Math 6620 Qualifying Exam, May 2022

Do **any one** of problems 1-2 and **any two** of problems 3-5. Clearly mark the problems you want to be graded. Each complete problem has equal value.

1) Suppose $f(x) \in C^{n+1}([a,b])$, and consider the interpolation problem: For $a \leq x_0 < x_1 < x_2 < ... < x_{n-1} < x_n \leq b$, find a polynomial $p(x)$ of degree at most n for which $p(x_j) = f(x_j)$, $j = 0, 1, ...n$.

(a) Show that this problem has at most one solution.

(b) Show that this problem has a solution.

(c) Show, for any $x \in [a,b]$, that

$$f(x) - p(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n),$$

for some $\xi \in [a,b]$.

2) Suppose you have a program for calculating approximations to a quantity A which takes a value of h as input and produces a value A_h as output. Suppose that

$$A = A_h + a_1 h + a_2 h^2 + a_3 h^3 + a_4 h^4 + O(h^5). \quad (1)$$

Here, h should be thought of as a small positive number, and a_1, a_2, a_3, and a_4 are fixed (but unknown) numbers.

(a) What is the order of the approximation A_h for A?

(b) Use Eq. (1) to derive an approximation B_h to A for which the error $A - B_h = O(h^3)$ and which uses only your program and some simple algebra.

3) Consider the 9-point discrete Laplacian

$$(\Delta_9^h U)_{j,l} = \frac{1}{6h^2}\Big(U_{j-1,l-1}+U_{j+1,l-1}+U_{j-1,l+1}+U_{j+1,l+1}+4U_{j,l-1}+4U_{j-1,l}+4U_{j+1,l}+4U_{j,l+1}-20U_{j,l}\Big)$$

(a) State and prove a maximum principle for $\Delta_9^h U$. (Hint: Think about what the key observation is in stating and proving a maximum principle for the standard 5-point discrete Laplacian.)

(b) Show that the following problem has a solution and that it is unique:

$$\begin{aligned} \frac{1}{6h^2}\Big(U_{j-1,l-1} + U_{j+1,l-1} + U_{j-1,l+1} + U_{j+1,l+1} + \\ 4U_{j,l-1} + 4U_{j-1,l} + 4U_{j+1,l} + 4U_{j,l+1} - 20U_{j,l}\Big) &= f_{j,l}, \end{aligned} \tag{2}$$

for $j = 1, ..., m$ and $l = 1, ...m$ (where $(m+1)\ h = 1$) along with the conditions $U_{j,l} = 0$ at grid points on the boundary of the domain.

(c) The local trucation error in using the scheme in Eq. (2) to find an approximate solution of $\Delta u = f$ for $0 < x < 1$ and $0 < y < 1$ with homogeneous Dirichlet boundary conditions is given by

$$\tau_{j,l} = \frac{1}{12}(u_{xxxx} + u_{xxyy} + u_{yyyy})h^2 + O(h^4),$$

where the derivatives of u are evaluated at $x_j = jh$, $y_l = lh$. Prove that the values $U_{j,l}$ obtained from Equations (2), along with zero discrete boundary values, converge to the values $u(x_j, y_l)$ of the true solution to the PDE BVP as $h \to 0$. (Hint: Use the relation between the set of global error values $E_{j,l} = U_{j,l} - u(x_j, y_l)$ and the values of the local truncation error.)

4) Suppose you are trying find an approximation $u(t)$ to a function $v(t)$ and you know that the following two equations hold:

$$\begin{aligned} u(t_{n+1}) &= C(k)u(t_n) \\ v(t_{n+1}) &= C(k)v(t_n) + k\tau(t_n) \end{aligned}$$

for some smooth function $\tau(t,k)$ that is not identically 0. Here, $k > 0$, $t_n = nk$, and $C(k)$ is a linear operator which depends on k but not on $u(t)$ or $v(t)$. State and prove conditions on $C(k)$ and $\tau(t,k)$ which are sufficient to imply that

$$\lim_{k \to 0, nk = t_n = t} u(t_n) = v(t).$$

5) For the initial value problem $u'(t) = f(u(t), t)$, $u(0) = \eta$, where $f(u,t)$ is continuous with respect to t and Lipschitz continuous with respect to u, consider the scheme

$$\frac{U^{n+2} - \frac{4}{3}U^{n+1} + \frac{1}{3}U^n}{k} = \frac{2}{3} f(U^{n+2}, t_{n+2}),$$

where $t_n = nk$, and U^n is supposed to approximate $u(t_n)$.

(a) Analyze the consistency, zero-stability, and convergence of this scheme.

(b) If you apply this scheme to the initial value problem

$$u'(t) = -10^{12}(u(t) - \cos(t)) - \sin(t), \qquad u(0) = 2,$$

what issues should you consider in choosing the timestep k? Are there time intervals in which k must be very small to get a reasonable solution and others in which it does not? Explain your answers.