UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Complex Analysis
May 25, 2022.

Instructions: This exam has two sections, A and B. Three (3) completely correct problems in section A will be a high pass and three (3) completely correct problems with at least one from section A is a passing exam. Note that solutions for problems in section B will not be counted towards a high pass. Be sure to provide all relevant definitions and statements of theorems cited. All solutions must include rigorous justification unless otherwise indicated.

Section A:

1. Let Ω be a simply connected domain in $\mathbb{C}$. Show that if a holomorphic function $f : \Omega \to \mathbb{C}$ has finitely many zeros, all of even order, then f has a holomorphic square root in Ω, i.e. there is a holomorphic function $g : \Omega \to \mathbb{C}$ so that $f(z) = g(z)^2$.

2. Show that if f is holomorphic on $\mathbb{D} \setminus \{0\}$ and
$$\lim_{z \to 0} z f(z) = 0$$
then f can be extended to a holomorphic function on $\mathbb{D}$.

3. Show that there is no function f holomorphic on $\mathbb{D}$ so that
$$f\left(\frac{1}{n}\right) = \frac{1}{n+2}$$
for all large $n \in \mathbb{N}$.

4. Suppose $f : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ is holomorphic and has an essential singularity at 0. Show that for all $\varepsilon > 0$ there is z near 0 so that
$$\left| f(z) - \frac{1}{z^4} \right| \leq \varepsilon.$$

5. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and zero at points $z_1, \ldots, z_n \in \mathbb{D}$ counted with multiplicity. Show that
$$|f(z)| \leq |\psi_{z_1}(z)| \cdots |\psi_{z_n}(z)| \text{ in } \mathbb{D}$$
where
$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

6. Using the residue calculus evaluate the integral
$$\int_{-\infty}^{\infty} \frac{1}{1 + x^4} dx$$

Section B:

7. Prove or disprove: There is no nonconstant entire function satisfying
$$\mathrm{Re}(f(z)) < \mathrm{Im}(f(z))$$
for all z. In your solution don't quote either Picard theorem.

8. State and prove Morera's Theorem.