DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
PH.D. PRELIMINARY EXAMINATION IN COMPLEX ANALYSIS

MAY 28, 2021

Instructions: This exam has two sections, A and B. Three (3) completely correct problems in section A will be a high pass and three (3) completely correct problems with at least one from section A is a passing exam. Note that solutions for problems in section B will not be counted towards a high pass. Be sure to provide all relevant definitions and statements of theorems cited. All solutions must include rigorous justification unless otherwise indicated.

Section A:

1. Show that any function which is meromorphic on the Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, is either constant or can be written as the sum of the singular parts of its Laurent series at its poles, in particular conclude that the function is rational.
2. Let U be a domain in $\mathbb{C}$. Consider the following collection of functions
$$X = \{f : U \to \mathbb{C} : \quad f \text{ holomorphic and } \textstyle\int_U |f|^2 \leq M\}$$
where $\int_U$ is the standard Lebesgue area integral. Show that X is compact with respect to the topology of uniform convergence on compact subsets of U (Note: this topology is metrizable so you can show sequential compactness).
3. Show that if $f(z)$ is entire and nowhere zero then there is an entire function $g(z)$ such that $f(z) = g(z)^2$.
4. Suppose that f_n are holomorphic on a domain $U \subset \mathbb{C}$ and $f_n(U) \subset \mathbb{C} \setminus \{w\}$ for all n and some fixed $w \in \mathbb{C}$. Show that if $f_n \to f$ locally uniformly in U then either $f(z) \equiv w$ or $f(U) \subset \mathbb{C} \setminus \{w\}$.
5. Suppose that f is entire and $f(\mathbb{C}) \subset \mathbb{C} \setminus [0, \infty)$, show that f is constant. Do not use either Picard Theorem (at least not without proof).
6. Compute the following integral using the techniques of complex analysis
$$\int_0^\infty \frac{\log(x)}{2 + x^2}\, dx.$$

Section B:

1. Suppose that f is holomorphic on the upper half plane $\mathbb{H} = \{\mathrm{Im}(z) > 0\}$, f extends continuously to $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$, and $f(\mathbb{R}) \subset \mathbb{R}$. Show that f has a unique holomorphic extension to $\mathbb{C}$ and explicitly identify that extension (in terms of f).
2. How many zeros does the function
$$f(z) = z^3 + 2 - e^{-z}$$
have in the right half plane $\{\mathrm{Re}(z) > 0\}$.