UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Differentiable Manifolds
January, 2023.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points.

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1. Define $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $F(x,y,z) = (x^2+y^2-2, z)$ and let $S^1 = \{(x,y) \in \mathbb{R}^2 | x^2+y^2=1\}$ be the unit circle. Show that $F^{-1}(S^1)$ is a smooth manifold and calculate its dimension.

2. Let V be a smooth vector field on $\mathbb{R}^2$ and assume that outside of a compact set $V = \frac{\partial}{\partial x}$ where $\frac{\partial}{\partial x}$ is the standard horizontal vector field. Show that the flow for V exists for all time.

3. Let M and N be submanifolds of $\mathbb{R}^n$ and for all $a \in \mathbb{R}^n$ let $M_a = \{x+a \in \mathbb{R}^n | x \in M\}$. Show that M_a and N are transverse for almost all choices of $a \in \mathbb{R}^n$.

4. Find two 1-forms α and β on $\mathbb{R}^2 - \{(0,0),(0,2)\}$ that are closed but not exact and have the property that $\alpha + t\beta$ is not exact for any $t \in \mathbb{R}$. (**Hint:** Let $\iota : S^1 \hookrightarrow \mathbb{R}^2$ be inclusion of S^1 into $\mathbb{R}^2$. Then choose α and β such that
$$\int_{S^1} \iota^*(\alpha + t\beta) = \int_{S^1} \iota^*\alpha \neq 0.)$$

5. Let M be a smooth manifold and TM its tangent bundle. Show that TM is orientable.

6. Let G be an n-dimensional Lie group. Show that the tangent bundle TG is isomorphic as a bundle to the product bundle $G \times \mathbb{R}^n$.