University of Utah, Department of Mathematics
January 2023, Algebra Qualifying Exam

There are five problems on the exam. You may attempt as many problems as you wish; three correct solutions count as a high pass. 2 correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.

1. Let $k = \mathbb{Z}/3\mathbb{Z}$ be the field with three elements. Find all the prime ideals in the ring $R = k[x,y]/(x^3 - x, y^2 + x)$.

2. Let $k = \mathbb{Z}/2\mathbb{Z}$ and $R = k[x]$. Let M denote the cokernel of the map $R^3 \longrightarrow R^3$ given by the matrix:

$$\begin{bmatrix} 1+x & x^2 & 1+x \\ 1 & x & 1 \\ 1 & x^2 & 1+x^2 \end{bmatrix}$$

   Write $\mathrm{Hom}(M, R/(x))$ as a direct sum of cyclic modules.

3. Let $R = (\mathbb{Z}/3\mathbb{Z})[x,y]$ and let $I = (x, 2y)$. Compute the number of elements in $\mathrm{Ext}^i(R/I, R)$ for each $i \geq 0$.

4. Suppose R is a commutative ring and I is an ideal. Let

$$J = \{x \in R \mid x^n \in I \text{ for some integer } n > 0\}.$$

   Prove directly that J is an ideal of R.

5. Determine up to similarity, all *real* 5×5-matrices with characteristic polynomial $x(x^2+1)^2$.