UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Real Analysis
January, 2023.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

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1. Let $(B, \|\cdot\|)$ be a Banach space. Explain the inclusion of B into its double dual. Prove that it is an isometry onto the image.

2. Let X be a set and $\mathcal{A}$ be an algebra on X. Let $\rho_0 : \mathcal{A} \to [0, \infty]$ be a pre-measure on $\mathcal{A}$. Recall that

$$\rho^*(S) = \inf\{\sum_{i=1}^{\infty} \rho_0(A_i) : A_i \in \mathcal{A} \text{ for all } i \text{ and } S \subset \cup_{i=1}^{\infty} A_i\}$$

is an outer measure. Define what it means that ρ_0 is a pre-measure and show that if $A \in \mathcal{A}$ then $\rho^*(A) = \rho_0(A)$.

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces and A be measurable with respect to $\mathcal{M} \times \mathcal{N}$, the σ-algebra generated by measurable rectangles. Show that if $A_1 \subset A_2 \subset ....$ satisfy that

$$(\mu \otimes \nu)(A_i) = \int_X \nu(\{y : (x, y) \in A_i\}) d\mu(x)$$

for all i then

$$(\mu \otimes \nu)(\cup_{i=1}^{\infty} A_i) = \int_X \nu(\{y : (x, y) \in \cup_{i=1}^{\infty} A_i\}) d\mu(x).$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space. Give an example of $f_1, ... : X \to \mathbb{C}$, measurable, which converge in measure to the zero function but do not converge μ-a.e. Show that there exists a subsequence $n_1, n_2, ...$ so that $f_{n_1}, ...$ converges almost everywhere to the zero function.

5. In this problem, let λ denote Lebesgue measure on $\mathbb{R}$ and $\hat{}$ denote the Fourier transform (on $\mathbb{R}$). Show that if f, g are Lebesgue integrable and $\hat{f} = \hat{g}$ then $f = g$ λ-a.e.

6. Assume that $\mu : \mathcal{B}_{\mathbb{R}^4} \to [0, \infty]$ is a measure which is outer regular and σ-finite. Show that if μ is inner regular on open sets then it is inner regular on all Borel sets. Recall that a Borel measure on $\mathbb{R}^4$ is called outer regular if for any $A \in \mathcal{B}_{\mathbb{R}^4}$ we have

$$\mu(A) = \inf\{\mu(U) : U \text{ is open and } A \subset U\}$$

and it is called inner regular if

$$\mu(A) = \sup\{\mu(K) : K \text{ is compact and } K \subset A\}.$$