Probability Qualifying Exam

January 2023

Instructions (Read before you begin)

- You may attempt all 6 problems in this exam. However, you can turn in solutions for **at most** 4 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is 10 points; a pass is 30 points or higher; a high pass is 36 points or higher.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Without using the Central Limit Theorem or the Law of the Iterated Logarithm, prove the Weak Law of Large Numbers when $X_i \in L^1$, i.e. show that when $X_1, X_2, \dots$ is a sequence of i.i.d. variables with $\mathbb{E}[X_1] = \mu$ and $X_1 \in L^1$, then

$$\bar{X} = \frac{X_1 + \ldots + X_n}{n} \to \mu \quad \text{in } L^1 .$$

   Hint: first, prove the claim when $X_1 \in L^2$. Then for $a > 0$, consider $X_i^a = X_i \cdot \mathbb{1}\{|X_i| \leq a\}$.

2. Let $\{A_i\}_{i=1}^\infty$ be a sequence of independent events such that $\sum_{i=1}^\infty \mathbb{P}(A_i) = \infty$. Prove that the A_n occur infinitely often with probability 1 (the converse of the Borel-Cantelli Lemma), i.e. that $\sum_{i=1}^\infty 1_{A_i} = \infty$ almost surely.

3. Show that if $X_n \to X$ in probability, then $X_n \Rightarrow X$, i.e. X_n converges weakly to X.

4. Recall the inversion formula: if μ is a probability measure and $\widehat{\mu}$ is its characteristic function, then for all $a < b$:

$$\lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \int_a^b e^{-ity}\, \widehat{\mu}(t)\; dy\; dt = \mu(a, b) + \frac{1}{2}\mu(\{a, b\})\, .$$

(a) Prove that if $\widehat{\mu} \in L^1$, then $\mu(\{a\}) = 0$ for all $a \in \mathbb{R}$ and μ has a probability density function given by

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity}\, \widehat{\mu}(t)\ dt\,.$$

(b) Show that if $X_1, \ldots, X_n$ are independent and uniformly distributed on $(-1, 1)$, then for $n \geq 2$, $X_1 + \ldots + X_n$ has density

$$f(y) = \frac{1}{\pi} \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^n \cos(ty)\ dt\,.$$

5. Prove that if $\{X_i\}_{i=1}^{\infty}$ are i.i.d. Uniform-[0,1] random variables, then

$$\frac{4\sum_{i=1}^{n} iX_i - n^2}{n^{\frac{3}{2}}}$$

converges weakly, and identify the limiting distribution. (Recall $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.)

6. Let X_n, Y_n be positive, in L^1, and measurable with respect to the filtration $\mathcal{F}_n$. Suppose

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq (1 + Y_n)X_n$$

with $\sum Y_n < \infty$ a.s.

(a) Show that $\prod_{i=1}^{n}(1 + Y_i)$ converges a.s. to a finite limit.

(b) Show

$$M_n = \frac{X_n}{\prod_{i=0}^{n-1}(1 + Y_i)}$$

is a super-martingale.

(c) Use (a) and (b) to prove that X_n converges a.s. to a finite limit.