Qualifying Exam
Analysis of Numerical Methods I, January 2022

Instructions: This exam is closed book, no notes, and no electronic devices or calculators are allowed. You have two hours and you will be graded on work for only 3 out of the 4 questions below. All questions have equal weight and a cumulative score of 65% or more on your 3 graded questions is considered a pass. A cumulative score of 80% or greater is considered a high pass. Indicate clearly the work and the 3 questions that you wish to be graded.

Problem 1. (Rank-One Perturbation of the Identity).
If \( u \) and \( v \) are \( n \)-vectors, the matrix \( B = I + uv^* \) is known as a rank-one perturbation of the identity. Show that if \( B \) is nonsingular, then its inverse has the form \( B^{-1} = I + \beta uv^* \) for some scalar \( \beta \), and give an expression for \( \beta \).
For what \( u \) and \( v \) is \( B \) singular? If it is singular, what is \( \text{null}(B) \)?

Problem 2. (Properties via SVD).
a) Consider \( A \in \mathbb{C}^{m \times n} \). Define what we mean by the singular value decomposition of \( A \).
b) Show that the rank of \( A \) is \( r \), the number of nonzero singular values.
c) Show that the largest singular value of \( A \), call it here \( \sigma_{\text{max}}(A) \), satisfies the relation
\[
\sigma_{\text{max}}(A) = \max_{y \in \mathbb{C}^m, x \in \mathbb{C}^n} \frac{|y^*Ax|}{||x||_2||y||_2}.
\]

Problem 3. (Properties of Projectors).
Prove algebraically:
a) Show that if \( P \in \mathbb{C}^{m \times m} \) is a nonzero projector, then \( ||P||_2 \geq 1 \).
b) Show that if \( P \) is an orthogonal projector, then \( I - 2P \) is unitary.

Problem 4. (Hadamard’s Inequality).
Prove algebraically:
Let \( A \in \mathbb{C}^{m \times m} \) and let \( a_j \) denote the \( j^{th} \) column of \( A \). Then, show that,
\[
|\det(A)| \leq \Pi_{j=1}^m ||a_j||_2.
\]