UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Differentiable Manifolds
Jan 7, 2022.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

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1. Let $X \subset \mathbb{R}^3$ be a smoothly embedded torus. Show that there is a plane $P \subset \mathbb{R}^3$ such that $X \cap P$ is a nonempty collection of circles.

2. Recall that the orthogonal group $O(n)$ is
$$O(n) = \{M \in \mathcal{M}_{n\times n} \mid MM^\top = I\}$$
where I is the identity matrix. Identifying the set $\mathcal{M}_{n\times n}$ of all real $n \times n$ matrices with $\mathbb{R}^{n^2}$, show that $O(n)$ is a submanifold of $\mathcal{M}_{n\times n}$ and compute the tangent space to $O(n)$ at I as a linear subspace of $T_I\mathcal{M}_{n\times n}$, which is identified with $\mathcal{M}_{n\times n}$.

3. In this problem you are allowed to use the fact that every closed 1-form on the 2-sphere S^2 is exact and every 2-form ω on S^2 such that $\int_{S^2} \omega = 0$ is exact. You are also allowed to use that $\int_{S^2} a^*(\omega) = -\int_{S^2} \omega$ for every 2-form ω on S^2, where $a : S^2 \to S^2$ is the antipodal map. Consider the map $p : S^2 \to \mathbb{R}P^2$ that identifies the antipodal points. Show that all closed 1- and 2-forms on $\mathbb{R}P^2$ are exact by considering their pullbacks to S^2 and applying the above facts.

4. Find an explicit closed 1-form ω on $\mathbb{R}^2 \setminus \{0\}$ which is not exact, and prove both statements.

5. Find a nonintegrable plane field in $\mathbb{R}^3$, with a proof.

6. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function such that $f(0) = 0$. Show that the graph of f
$$Gr(f) = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^n\}$$
and the diagonal
$$\Delta = \{(x, x) \in \mathbb{R}^n \times \mathbb{R}^n\}$$

intersect transversely at $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^n$ if and only if the derivative $f_*(0) : T_0\mathbb{R}^n \to T_0\mathbb{R}^n$ does not have $+1$ as an eigenvalue.