University of Utah, Department of Mathematics
Algebra 1 Qualifying Exam
January 2022

There are five problems on this exam. You may attempt as many problems as you wish; two correct solutions count as a *pass*, and three correct as a *high pass*. Show all your work, and provide reasonable justification for your answers.

1. Consider the ideal $I := (2x-9,\ 3x-7)$ in the ring $\mathbb{Z}[x]$. Find the smallest positive integer n such that
$$(x^{26}+x+1)^{13}-n$$
belongs to the ideal I.

2. Let R be a commutative ring with identity, and let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules. Prove or disprove:
   (a) If M and N are finitely generated R-modules, then L is finitely generated.
   (b) If M is finitely generated, and N is a free R-module, then L is finitely generated.

3. Let $R := \mathbb{Q}[x]/(x^3-1)$. Give an example of a finitely generated projective R-module that is not free.

4. Let R be a commutative ring with identity such that $IJ = I \cap J$ for all ideals I and J. Prove that each prime ideal of R is maximal.

5. Let M be a 5×5 matrix over the complex numbers $\mathbb{C}$, such that the eigenvectors of M, along with the zero vector, form a two-dimensional vector subspace of $\mathbb{C}^5$. Determine the possible Jordan forms of M.