Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve completely at least three problems and score at least 30 points. For a pass you need to solve completely at least two problems and score at least 25 points. Carefully state any theorems you use.

1. Prove the Borel-Cantelli lemma: Let $(X, M, \mu)$ be a measure space and let $A_i$ be measurable sets such that $\sum_{i=1}^{\infty} \mu(A_i) < \infty$. Then the set
   \[ \Omega = \{ x \in X \mid x \in A_i \text{ for infinitely many } i \} \]
of points that belong to infinitely many $A_i$ has measure 0.

2. Let $X$ be a compact metric space, $M$ the Borel $\sigma$-algebra on $X$, and $\mu$ a finite measure on $M$ (meaning that $\mu(X) < \infty$; we call such a measure a finite Borel measure on $X$). Prove that $\mu$ is regular, i.e. for every Borel set $E \subset X$
   \begin{itemize}
   \item $\mu(E) = \sup\{ \mu(K) \mid K \subseteq E \text{ is compact} \}$, and
   \item $\mu(E) = \inf\{ \mu(U) \mid U \supseteq E \text{ is open} \}$.
   \end{itemize}
   Hint: Show that the set of $E$’s that satisfy both bullets is a $\sigma$-algebra that contains compact sets.

3. (a) Let $(X, M)$ be a measurable space, $\mu$, $\nu$ positive measures on $(X, M)$. Suppose $\nu(X) < \infty$. Prove that $\nu \ll \mu$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu(E) < \epsilon$.
   
   (b) For $\mu$ the Lebesgue measure on $\mathbb{R}$ find a positive (but infinite) measure $\nu$ such that $\nu \ll \mu$ but the $\epsilon - \delta$ statement in (a) fails.

4. Let $(X, M, \mu)$ be a measure space with $\mu(X) < \infty$. Let $f : X \rightarrow X$ be a measure preserving bijection, in the sense that $\mu(A) = \mu(f(A))$ for every $A \in M$. Prove that for every $E \in M$
   \[ \{ x \in E \mid f^n(x) \notin E \text{ for all } n > 0 \} \]
has measure 0.
5. Let $V, W$ be Banach spaces, $T_i : V \to W$ a sequence of bounded linear operators such that $\lim_{i \to \infty} T_v$ exists for every $v \in V$. Define the linear map $T : V \to W$ by

$$ T_v = \lim_{i \to \infty} T_v $$

Show that $T$ is bounded.

6. Show that the Banach space $\ell^p$ with the usual norm is not a Hilbert space when $p \in [1, 2) \cup (2, \infty)$. 