Probability Qualifying Exam

January 2022

Instructions (Read before you begin)

- You may attempt all 7 problems in this exam. However, you can turn in solutions for **at most** 4 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is 10 points; a pass is 30 points or higher; a high pass is 36 points or higher.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Suppose $\{r_k\}_{k\in\mathbb{Z}_+}$ is a sequence of real numbers such that $n^{-1}\sum_{k=1}^{n}|r_k| \to 0$ as $k\to\infty$. Let $\{X_k\}_{k\in\mathbb{N}}$ be a sequence of L^2 random variables such that $\mathrm{E}[X_k]=0$ and $\mathrm{E}[X_kX_\ell] \leqslant r_{\ell-k}$ for all $k\leqslant \ell$. Let $S_n = X_1+\cdots+X_n$. Show that $n^{-1}S_n$ converges in L^2 as $n\to\infty$ and identify the limit.

2. Answer the following two questions.

   (a) Let $\{X_n\}_{n\in\mathbb{N}}$ be independent, almost surely finite random variables. Show that
   $$\sup_n X_n < \infty \text{ a.s. if, and only if, } \exists c<\infty \text{ such that } \sum_n \mathrm{P}(X_n > c) < \infty.$$

   (b) Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of events with $\mathrm{P}(A_n) \geqslant \delta$ for each n. Show that
   $$\mathrm{P}(A_n \text{ happens infinitely often }) \geqslant \delta.$$

3. Consider a probability space $(\Omega,\mathcal{F},\mathrm{P})$. Let $X\in L^1$. Suppose $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is an increasing sequence of σ-algebras such that $\cup_{n\in\mathbb{N}}\mathcal{F}_n$ generates $\mathcal{F}$. Prove the Doob martingale convergence theorem that states that $\mathrm{E}[X\mid\mathcal{F}_n]$ converges almost surely to X.

4. Let u be an integrable Borel function on $[0,1)$ relative to the Lebesgue measure. For each $n\geqslant 1$ and $x\in[0,1)$, let $I_n(x) = [k2^{-n},(k+1)2^{-n})$ be the interval that contains x as k varies from 0 to 2^n-1. Show that for Lebesgue-a.e. x,
   $$\lim_{n\to\infty} 2^n \int_{I_n(x)} u(y)\,dy = u(x).$$
   (Hint: Cast this in a suitable martingale framework then use the result of problem 3.)

5. Fix an integer $n \geqslant 1$. Let $\{Y_k\}_{k\in\mathbb{N}}$ be i.i.d. uniform on $\{1, 2, ..., n\}$. Let

$$\sigma_n = \inf\{k \geqslant 2 : Y_k = Y_m \text{ for some } 1 \leqslant m < k\}$$

be the first time we get a repeated sample. Find an exponent γ and a nondegenerate distribution μ such that, as $n \to \infty$, $n^{-\gamma}\sigma_n$ converges weakly to the distribution μ. (Hint: look at tail probabilities $\mathrm{P}(n^{-\gamma}\sigma_n \geqslant x)$.)

6. Give an example of a sequence of independent random variables $\{Z_k\}_{k\in\mathbb{N}}$ satisfying $\mathrm{E}[Z_k] = 0$, $\mathrm{Var}(Z_k) = 1$, for each k, and such that $(Z_1 + \cdots + Z_n)/\sqrt{n}$ does not converge in distribution to a standard normal. Justify any claims you make.

7. Let $\{X_n\}_{n\in\mathbb{N}}$ be independent random variables such that for each n, X_n takes the values $+1$ and -1 with probability $(1 - 2^{-n})/2$ each and the value 2^k with probability 2^{-k}, for integers $k > n$.

   (a) Show that $\mathrm{E}[|X_n|] = \infty$ for all $n \in \mathbb{N}$.

   (b) Show that $(X_1 + \cdots + X_n)/\sqrt{n}$ converges in distribution to a standard normal. (Hint: Check that we can write $X_n = (1 - B_n)Y_n + 2^n B_n Z_n$, where B_n is Bernoulli(2^{-n}), Y_n takes values ± 1 equally likely, Z_n takes the value 2^k with probability 2^{-k}, for $k \in \mathbb{N}$, and $\{B_n, Z_n, Y_n : n \geqslant 1\}$ are mutually independent. Now, X_n is a small perturbation of $(1 - B_n)Y_n$. Work out the details.)