Instructions (Read before you begin)

- You may attempt all 7 problems in this exam. However, you can turn in solutions for at most 4 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is 10 points; a pass is 30 points or higher; a high pass is 36 points or higher.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Suppose \( \{r_k\}_{k \in \mathbb{Z}^+} \) is a sequence of real numbers such that \( n^{-1} \sum_{k=1}^{n} |r_k| \to 0 \) as \( k \to \infty \).
   Let \( \{X_k\}_{k \in \mathbb{N}} \) be a sequence of \( L^2 \) random variables such that \( E[X_k] = 0 \) and \( E[X_kX_\ell] \leq r_{\ell-k} \) for all \( k \leq \ell \). Let \( S_n = X_1 + \cdots + X_n \). Show that \( n^{-1}S_n \) converges in \( L^2 \) as \( n \to \infty \) and identify the limit.

2. Answer the following two questions.
   (a) Let \( \{X_n\}_{n \in \mathbb{N}} \) be independent, almost surely finite random variables. Show that \( \sup X_n < \infty \) a.s. if, and only if, \( \exists c < \infty \) such that \( \sum P(X_n > c) < \infty \).
   (b) Let \( \{A_n\}_{n \in \mathbb{N}} \) be a sequence of events with \( P(A_n) \geq \delta \) for each \( n \). Show that \( P(A_n \text{ happens infinitely often}) \geq \delta \).

3. Consider a probability space \((\Omega, \mathcal{F}, P)\). Let \( X \in L^1 \). Suppose \( (\mathcal{F}_n)_{n \in \mathbb{N}} \) is an increasing sequence of \( \sigma \)-algebras such that \( \cup_{n \in \mathbb{N}} \mathcal{F}_n \) generates \( \mathcal{F} \). Prove the Doob martingale convergence theorem that states that \( E[X | \mathcal{F}_n] \) converges almost surely to \( X \).

4. Let \( u \) be an integrable Borel function on \([0,1)\) relative to the Lebesgue measure. For each \( n \geq 1 \) and \( x \in [0,1) \), let \( I_n(x) = [k2^{-n}, (k+1)2^{-n}) \) be the interval that contains \( x \) as \( k \) varies from 0 to \( 2^n - 1 \). Show that for Lebesgue-a.e. \( x \),
\[
\lim_{n \to \infty} 2^n \int_{I_n(x)} u(y) \, dy = u(x).
\]
(Hint: Cast this in a suitable martingale framework then use the result of problem 3.)
5. Fix an integer \( n \geq 1 \). Let \( \{ Y_k \}_{k \in \mathbb{N}} \) be i.i.d. uniform on \( \{1, 2, \ldots, n\} \). Let

\[
\sigma_n = \inf\{ k \geq 2 : Y_k = Y_m \text{ for some } 1 \leq m < k \}
\]

be the first time we get a repeated sample. Find an exponent \( \gamma \) and a nondegenerate distribution \( \mu \) such that, as \( n \to \infty \), \( n^{-\gamma} \sigma_n \) converges weakly to the distribution \( \mu \).
(Hint: look at tail probabilities \( \Pr(n^{-\gamma} \sigma_n \geq x) \).

6. Give an example of a sequence of independent random variables \( \{ Z_k \}_{k \in \mathbb{N}} \) satisfying \( \mathbb{E}[Z_k] = 0 \), \( \text{Var}(Z_k) = 1 \), for each \( k \), and such that \( (Z_1 + \cdots + Z_n)/\sqrt{n} \) does not converge in distribution to a standard normal. Justify any claims you make.

7. Let \( \{ X_n \}_{n \in \mathbb{N}} \) be independent random variables such that for each \( n \), \( X_n \) takes the values \(+1\) and \(-1\) with probability \((1 - 2^{-n})/2\) each and the value \(2^k\) with probability \(2^{-k}\), for integers \( k > n \).

(a) Show that \( \mathbb{E}[|X_n|] = \infty \) for all \( n \in \mathbb{N} \).

(b) Show that \( (X_1 + \cdots + X_n)/\sqrt{n} \) converges in distribution to a standard normal.
(Hint: Check that we can write \( X_n = (1 - B_n)Y_n + 2^n B_n Z_n \), where \( B_n \) is Bernoulli(\(2^{-n}\)), \( Y_n \) takes values \( \pm 1 \) equally likely, \( Z_n \) takes the value \(2^k\) with probability \(2^{-k}\), for \( k \in \mathbb{N} \), and \( \{B_n, Z_n, Y_n : n \geq 1\} \) are mutually independent. Now, \( X_n \) is a small perturbation of \((1 - B_n)Y_n\). Work out the details.)