UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. preliminary Examination on
Applied Linear Operators and Spectral Methods (Math 6710)
January 2021

Instructions: This examination includes five problems but you are to work three of them. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three problems will be graded. All problems are worth 20 points. A pass is 35 or more points. A high-pass is 45 or more points.

1. Let H be a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$. Let T be a bounded linear operator satisfying $\langle Tx, x \rangle \geq 0$ for all $x \in H$. Show that $I + T$ is injective.

2. Let $T \colon H \to H$ be a bounded linear operator on a Hilbert space H. Let A and B be subsets of H such that $T(A) \subset B$. Show that $A^\perp \supset T^*(B^\perp)$.

   Note: We used the notation $T(A) = \{Tx \mid x \in A\}$.

3. Let X be a Banach space and $T \colon X \to X$ be a compact linear operator. Use the Fredholm alternative to show that if $\lambda \notin \sigma_p(T)$ and $\lambda \neq 0$ then $\lambda \in \rho(T)$, or in other words: that all non-zero spectral values of a compact operator are eigenvalues.

4. Let X be a Banach space and consider the **non-linear** mapping $F \colon X \to X$ defined by

   $$F(x) = y - \alpha \sin(\|x\|)x,$$

   where $y \in X$ is fixed and α is a scalar. Show that there is a constant $C > 0$ such that for any $|\alpha| < C$, F is a contraction on the open ball of radius 1 centered at y.

   Note: You may use that for any $a, b \in \mathbb{R}$ we have $|\sin(a) - \sin(b)| \leq |a - b|$.

5. Let $T \colon X \to X$ be a bounded linear operator on a complex Banach space X.

   Prove that $\sigma(T)$ lies in the complex plane disk: $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$.