UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Differentiable Manifolds
January, 2021.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points.

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1. View the 2-torus as
$$T^2 = \{(z, w) \in \mathbb{C}^2 \mid |z| = |w| = 1\}$$
and let $f : T^2 \to T^2$ be defined by
$$f(z, w) = (z^2, w^2)$$
Prove that f is not homotopic to the identity.

2. Let $M \subset \mathbb{R}^3$ be a submanifold diffeomorphic to the circle. Prove that for every $\epsilon > 0$ there is a vector $v \in \mathbb{R}^3$ of norm $< \epsilon$ such that M and $M + v = \{x + v \mid x \in M\}$ are disjoint. Carefully state any theorems you are citing.

3. Let ω and η be two differential forms on a manifold M.
   (a) Define the wedge product $\omega \wedge \eta$.
   (b) Prove that if ω and η are both closed then so is $\omega \wedge \eta$.
   (c) Prove that the de Rham cohomology class of $\omega \wedge \eta$ is unchanged if ω and η are replaced by different representatives of the same cohomology classes.

   You are allowed to use the identity relating exterior differentiation and wedge product, but you should state it carefully.

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the function $f(x, y, z) = (xy, z)$ and let $\omega = dx \wedge dy$ be the area form on $\mathbb{R}^2$. Compute the pull-back $f^*(\omega)$.

5. Define two vector fields V and W on $\mathbb{R}^3$ as follows: at $p = (a, b, c)$ let $V(p) = \frac{\partial}{\partial x} + b\frac{\partial}{\partial z}$ and $W(p) = \frac{\partial}{\partial y}$. Prove that there is no nonempty surface $S \subset \mathbb{R}^3$ such that both V and W are tangent to S at each of its points.

6. Let $\mathcal{M}(n)$ denote the space of all $n \times n$ matrices with real entries, identified with $\mathbb{R}^{n^2}$, and let $\mathcal{S}(n)$ be the subspace of symmetric matrices. Consider the smooth map $F : \mathcal{M}(n) \to \mathcal{S}(n)$ defined by $f(A) = AA^\top$, where $A^\top$ is the transpose of A.

   (a) Show that the identity matrix I is a regular value of f.

   (b) Deduce that the orthogonal group

   $$O(n) = \{A \in \mathcal{M}(n) \mid AA^\top = I\}$$

   is a submanifold of $\mathcal{M}(n)$. Carefully state any theorems you are citing.