University of Utah, Department of Mathematics
Algebra 1 Qualifying Exam
January 2021

There are five problems on this exam. You may attempt as many problems as you wish; two correct solutions count as a *pass*, and three correct as a *high pass*. Show all your work, and provide reasonable justification for your answers.

1. Let d be the greatest common divisor of positive integers n and m. Prove that

$$\mathbb{Z}/(n) \otimes_{\mathbb{Z}} \mathbb{Z}/(m) \;\cong\; \mathbb{Z}/(d).$$

2. Let M be the cokernel of the map

$$\mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 2 & 2 & 4 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \end{pmatrix}} \mathbb{Z}^3.$$

   Write M as a direct sum of cyclic $\mathbb{Z}$-modules.

3. Let I_k denote the $k \times k$ identity matrix. Suppose M is a $2n \times 2n$ matrix over $\mathbb{Q}$ such that $M^2 = -I_{2n}$, prove that M is similar to the matrix

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

4. State the Noether normalization lemma.

   Let M be a maximal ideal of the ring $A = \mathbb{R}[x,y,z]/(x^2+y^2+z^2+1)$. Determine the field A/M.

5. Consider the ring homomorphism given by

$$\begin{aligned} \mathbb{Z}[x] &\longrightarrow \quad \mathbb{Z} \times \mathbb{Z} \\ f(x) &\longmapsto \big(f(1),\ f(-1)\big) \end{aligned}$$

   Is this homomorphism surjective? Determine a minimal generating set for the kernel.