UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Differentiable Manifolds
August 18, 2020.

Instructions. This exam covers material from Math 6510. Not all questions are equally difficult. Answer five of the following questions. Indicate which of the questions are to be graded. If you do more than five, only the first five will be graded. Each question is worth 20 points. The score needed to pass is 55. The score needed for a high pass is 65.

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1. Let M be a connected manifold and $f : M \to N$ be smooth with $df_p(v) = 0$ for all $p \in M$ and $v \in T_pM$. Show that f is constant.

2. Show that the tangent bundle of standard two sphere $\mathbb{S}^2$ is not trivial.

3. Consider the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$\Phi(x, y, s, t) = \big(x^2 + y, \quad x^2 + y^2 + s^2 + t^2 + y\big).$$

   Show that $\Phi^{-1}(0, 1)$ is a properly embedded submanifold. What manifold is it?

4. If M and N are smooth manifolds such that N is non-orientable, prove that $M \times N$ is non-orientable.

5. Let S be the 3×3 diagonal matrix whose entries are $S = \operatorname{diag}\{1, 1, -2\}$. Let G denote the set of real 3×3 matrices that satisfy $A^T SA = S$, where A^T denotes the transpose.

   (a) Show that with the operation of matrix multiplication, G becomes a Lie group.

   (b) Show that the Lie algebra $\mathfrak{g}$ of G may be identified with the set of 3×3 matrices X that satisfy $X^T S + SX = 0$. What is the Lie algebra bracket?

   (c) If α is a nonzero real number, show that the matrix $X = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ lies in the Lie algebra $\mathfrak{g}$. What is the one parameter subgroup of G that is tangent to X at $t = 0$?

   (d) Let $X, Y \in \mathfrak{g}$ be two matrices in the Lie algebra that commute. Let Δ be the distribution obtained by translating the two dimensional vector space $\operatorname{span}\{X, Y\}$ around by the group. Is Δ integrable?

6. Let M be a smooth connected n-dimensional manifold, ω be a smooth closed one form on M and $p_0, p_1 \in M$. If $\gamma_s : [0, 1] \to M$ for $0 \le s \le 1$ is a smooth homotopic family of curves with $\gamma_s(0) = p_0$ and $\gamma_s(1) = p_1$ for all s, show that $\int_0^1 \gamma_s^* \omega$ is independent of s. Conclude that if M is also simply connected, then the first de Rham cohomology group vanishes $H^1_{\mathrm{dR}}(M) = \{0\}$.