UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Partial Differential Equations

August 18, 2020.

Instructions: This examination consists of working on four of the six given problems. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first four will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth [15] points. A high-passing score is [50] and a passing score is [40].

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1. Suppose that $\rho(x,t)$ is the number density of cars evolving according to the traffic model

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0.$$

with u the car speed. Take $u = 1 - \rho$. A queue is building up at a traffic light $x = 1$ so that, when the light turns green at $t = 0$,

$$\rho(x,0) = \begin{cases} 0, & x < 0 \text{ and } x > 1 \\ x, & 0 < x < 1. \end{cases}$$

   (a) Write down the characteristic equations.

   (b) Solve the characteristic equations and construct the corresponding characteristic diagram.

   (c) Deduce that a collision first occurs at $x = 1/2$ when $t = 1/2$, and that thereafter there is a shock $S(t)$. Use the Rankine-Hugonoit condition to show that

$$\frac{dS}{dt} = \frac{S + t - 1}{2t}.$$

2. A problem in the dynamics of the overhead power wire for an electric locomotive leads to the model

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad x \neq X(t),\, t > 0,$$

where X is a prescribed smooth function with $0 < X'(t) < 1$. Here $X(t)$ is the locomotive position and u is the displacement of the wire. Across $x = X$ there are prescribed discontinuities

$$\left[\frac{\partial u}{\partial x}\right]_{X^-}^{X^+} = -V(X(t),t), \quad [u]_{X^-}^{X^+} = 0.$$

Suppose that $u = u_t = 0$ at $t = 0$. By constructing a weak solution that takes into account the discontinuities across X, show that

$$u(x,t) = \int_0^{\tau_0} V(X(\tau),\tau)(1 - X'(\tau)^2)d\tau,$$

where the range of τ is taken so that $y = X(\tau)$ lies within the range

$$\tau - t < y - x < t - \tau, \; \tau > 0.$$

HINT: you will need to derive a relationship between discontinuities in $\partial_t u$ and in $\partial_x u$ along $X(t)$. You will also need to consider the domain of dependence in the (x,t)-plane.

3. Consider the following reaction-diffusion (RD) system on a bounded domain $\Omega \subset \mathbb{R}^d$:

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{D}\nabla^2 \mathbf{u} + \mathbf{f}(\mathbf{u}),$$

where $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{D} = \text{diag}(D_1, \ldots, D_n)$. Assume no-flux boundary conditions $(\mathbf{n} \cdot \nabla)\mathbf{u} = 0$ on $\partial\Omega$. Let us define the energy function

$$E(t) = \frac{1}{2}\int_\Omega \|\nabla \mathbf{u}\|^2 d\mathbf{x}, \quad \|\nabla \mathbf{u}\|^2 = \sum_{i=1}^n \nabla u_i \cdot \nabla u_i = \sum_{k=1}^d \sum_{i=1}^n \left(\frac{\partial u_i}{\partial x_k}\right)^2.$$

   (a) Differentiating $E(t)$ using integration by parts, the boundary conditions, and the RD equations, show that

$$\frac{dE}{dt} \leq -d\int_\Omega \|\nabla^2 \mathbf{u}\|^2 d\mathbf{x} + mE,$$

   where $d = \min_i D_i$ and

$$m = \max_{\mathbf{u}} \|\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u})\| = \max_{u_1,\ldots u_n} \sqrt{\sum_{i,j=1}^n \left(\frac{\partial f_i}{\partial u_j}\right)^2}.$$

   (b) Poincare's inequality states that for any function $v(\mathbf{x})$ in Ω that satisfies the Neumann boundary condition on $\partial\Omega$ and $\int_\Omega v(\mathbf{x})d\mathbf{x} = 0$, the following holds:

$$\int_\Omega v(\mathbf{x})^2 d\mathbf{x} \leq \frac{1}{\mu_1}\int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x},$$

   where μ_1 is the smallest nonzero eigenvalue of the Laplacian operator $-\nabla^2$. Use Poincare's inequality with $v(\mathbf{x}) = \partial_{\mathbf{x}_k} u_i(\mathbf{x})$, and then sum over $k = 1, \ldots, d$, $i = 1, \ldots, n$ to show that

$$\int_\Omega \|\nabla \mathbf{u}\|^2 d\mathbf{x} \leq \frac{1}{\mu_1}\int_\Omega \|\nabla^2 u(\mathbf{x})\|^2 d\mathbf{x},$$

   and thus

$$\frac{dE}{dt} \leq (m - 2\mu_1 d)E.$$

4. A function $u \in C^2(\Omega), \Omega \subset \mathbb{R}^n$, is subharmonic (superharmonic) in Ω if $\nabla^2 u \geq 0$ ($\nabla^2 u \leq 0$) in Ω.

   (a) By extending the derivation of mean value properties, show that if u is subharmonic then for every ball $B_R(\mathbf{x}) \subset\subset \mathbb{R}$

$$u(\mathbf{x}) \leq \frac{1}{\omega_n R^{n-1}}\int_{\partial B_R(\mathbf{x})} u(\sigma)d\sigma.$$

   Hence show that

$$u(\mathbf{x}) \leq \frac{n}{\omega_n R^n}\int_{B_R(\mathbf{x})} u(\mathbf{y})d\mathbf{y}.$$

   How are the inequalities changed if u is superharmonic?

(b) Show that if $u \in C(\overline{\Omega})$ is subharmonic (superharmonic), the maximum (minimum) of u is attained only on $\partial\Omega$, unless u is constant.

(c) Let u be subharmonic in Ω and $F : \mathbb{R} \to \mathbb{R}$ smooth. Under what conditions on F is $F(u)$ subharmonic?

5. Consider the diffusion equation in a spherical cell of radius R:

$$\frac{\partial u(\mathbf{x}, t)}{\partial t} = D\nabla^2 u(\mathbf{x}, t), \quad 0 < |\mathbf{x}| < R,$$

with boundary condition $u(|\mathbf{x}| = R, t) = u_1$ and initial condition $u(\mathbf{x}, 0) = u_0$ with u_0, u_1 constants.

(a) Assume a radially symmetric solution $v(r, t) = u(r, t) - u_1$ so that,

$$\frac{\partial v(r, t)}{\partial t} = D\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}D\frac{\partial v}{\partial r}, \quad 0 < r < R,$$

with $v(R, t) = 0$ and $v(r, 0) = u_0 - u_1$. Use separation of variables $v(r, t) = V(r)T(t)$ to derive the general solution

$$u(r, t) = \sum_{n=1}^{\infty} c_n \mathrm{e}^{-tDn^2\pi^2/R^2} \frac{1}{r} \sin(n\pi r/R) + u_1.$$

Hint: in order to solve the boundary value problem for $V(r)$, perform the change of variables $\hat{V}(r) = rV(r)$.

(b) Setting $t = 0$ in the general solution and using $v(r, 0) = u_0 - u_1$, determine the coefficients c_n. Hint: you will need to use the identity

$$\int_0^R \sin(n\pi r/R) \sin(m\pi r/R) dr = \frac{R}{2}\delta_{n,m}.$$

(c) Determine an approximation for the concentration $u(0, t)$ at the center of the sphere by taking the limit $r \to 0$, with $r^{-1}\sin(\theta r) \to \theta$. Keeping only the leading order exponential term ($n = 1$), show that the time τ for the center to reach a concentration u^*, $u_1 < u^* < u_0$, is approximately

$$\tau = \frac{R^2}{D\pi^2} \ln \frac{2(u_0 - u_1)}{u^* - u_1}.$$

6. Consider the following Dirichlet problem for the Laplacian in a two-dimensional half-space (x, y):

$$\nabla^2 u = 0, \; x \in \mathbb{R}, \; y > 0; \quad u(x, 0) = g(x).$$

(a) Use the fundamental solution and the method of images to explain why the corresponding Dirichlet Green's function is given by

$$G(x, y|x', y') = \frac{1}{2\pi} \log \frac{\sqrt{(x - x')^2 + (y - y')^2}}{\sqrt{(x - x')^2 + (y + y')^2}},$$

where

$$\nabla^2 G(x, y|x', y') = -\delta(x - x')\delta(y - y'), \; x \in \mathbb{R}, \; y > 0; \quad G(x, y|x', 0) = 0,$$

with differentiation with respect to the coordinates (x', y').

(b) Using Green's identity show that the solution of the Dirichlet problem can be written as

$$u(x,y) = -\int_{-\infty}^{\infty} \left. \frac{\partial G(x,y|x',y')}{\partial y'} \right|_{y'=0} g(x')dx'.$$

Hence, show that

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(x)}{(x'-x)^2 + y^2} dx'.$$