UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Ordinary Differential Equations

August 18, 2020.

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Instructions: This examination consists of working on four of the six given problems. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first four will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth [15] points. A high-passing score is [50] and a passing score is [40].

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1. Let $f \in C^1(U, \mathbb{R}^n)$ for $U \subset \mathbb{R}^n$ and $x_0 \in U$. Given the Banach space $X = C([0,T], \mathbb{R}^n)$ with norm $\|x\| = \max_{0 \le t \le T} |x(t)|$, let

$$K(x)(t) = x_0 + \int_0^t f(x(s))ds$$

for $x \in X$. Define $V = \{x \in X \mid \|x - x_0\| \le \epsilon\}$ for fixed $\epsilon > 0$ and suppose $K(x) \in V$ (which holds for sufficiently small $T$), so that $K : V \to V$ with $V$ a closed subset of $X$.

   (a) Using the fact that $f$ is locally Lipschitz in $U$ with Lipschitz constant $L_0$, and taking $x, y \in V$ show that
   $$|K(x(t)) - K(y(t))| \le L_0 t \|x - y\|.$$
   Hence, show that $K$ satisfies the contraction mapping principle
   $$\|K(x) - K(y)\| \le L_0 T \| |x - y\| \quad x, y \in V.$$

   (b) Choosing $T < 1/L_0$, apply the contraction mapping principle to show that the integral equation has a unique continuous solution $x(t)$ for all $t \in [0,T]$ and sufficiently small $T$. Hence establish existence and uniqueness of the initial value problem
   $$\frac{dx}{dt} = f(x), \quad x(0) = x_0.$$

2. (a) Consider the dynamical system
   $$\begin{aligned} \dot{x} &= -y + x(1 - z^2 - x^2 - y^2) \\ \dot{y} &= x + y(1 - z^2 - x^2 - y^2) \\ \dot{z} &= 0. \end{aligned}$$
   Determine the invariant sets and attracting set of the system. Determine the $\omega$–limit set of any trajectory for which $|z(0)| < 1$. Also sketch the flow.

   (b) Use the Poincare$'$-Bendixson (PB) Theorem and the fact that the planar system
   $$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$
   has only the one critical point at the origin to show that this system has a periodic orbit in the annular region $A = \{x \in R^2 \mid 1 < |x| < \sqrt{2}\}$. HINT: Convert to polar coordinates.

3. (a) Consider the equation

$$\frac{d\mathbf{x}}{dt} = f(t)\mathbf{A}\mathbf{x},\ \ \mathbf{x} \in \mathbf{R}^2$$

with $f(t)$ a scalar $T$–periodic function and $\mathbf{A}$ a constant matrix with real distinct eigenvalues $\lambda_1, \lambda_2$. Show that the corresponding Floquet multipliers are

$$\mu_i = \exp\left(\lambda_i \int_0^T f(s)ds\right),\ \ i = 1, 2.$$

(b) Consider the Hill equation

$$\ddot{x} + q(t)x = 0, \quad q(t+T) = q(t).$$

Rewrite as a first-order system. Use Liouiville's formula to show that the characteristic multipliers have the form

$$\mu_{\pm} = \Delta \pm \sqrt{\Delta^2 - 1},$$

where $\Delta = \mathrm{tr}(\Psi(T))/2$ and $\Psi(t)$ is the fundamental matrix with $\Psi(0) = \mathbf{I}$. Hence, show that all solutions are bounded if $|\Delta| < 1$.

4. Consider the following linear equation for $\mathbf{x} \in \mathbb{R}^n$:

$$\dot{\mathbf{x}} = \mathbf{A} + \mathbf{B}(\mathbf{t})\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

where $\mathbf{A}, \mathbf{B}(t)$ are $n \times n$ matrices. Suppose that all eigenvalues $\lambda_j$, $j = 1, \ldots, n$, of the matrix $\mathbf{A}$ satisfy $\mathrm{Re}(\lambda_j) < 0$, and let $\mathbf{B}(t)$ be continuous for $0 \leq t < \infty$ with $\int_0^\infty \|\mathbf{B}(t)\|dt < \infty$.

(a) Using the variation of constants formula show that there exist constants $K, \sigma > 0$ such that

$$|\mathbf{x}(t)| \leq K\mathrm{e}^{-\sigma(t-t_0)}|\mathbf{x}_0| + K\int_{t_0}^t \mathrm{e}^{-\sigma(t-s)}|\mathbf{x}(s)|\|\mathbf{B}(s)\|ds.$$

(b) Let $u(t) = \mathrm{e}^{\sigma t}|\mathbf{x}(t)|$, $v(t) = \|\mathbf{B}(t)\|$ and $c = K\mathrm{e}^{\sigma t_0}|\mathbf{x}_0|$. Show that the inequality of part (a) can be rewritten as

$$u(t) \leq c + \int_{t_0}^t v(s)u(s)ds.$$

(c) From Gronwall's inequality we have

$$u(t) \leq c\exp\left(\int_{t_0}^t v(s)ds\right).$$

Use this to show that the zero solution of the IVP is asymptotically stable.

5. Consider the system

$$\dot{u} = v, \quad \dot{v} = -v + \mu u - u^2.$$

(a) Show that the linearized system at $\mu = 0$ can be diagonalized under the transformation $x = u + v$ and $y = -v$. Show that in these new coordinates the full system becomes

$$\dot{x} = \mu(x+y) - (x+y)^2, \quad \dot{y} = -y - \mu(x+y) + (x+y)^2.$$

(b) Now consider the extended system in which we add the equation

$$\dot{\mu} = 0.$$

Show that the extended center manifold in a neighborhood of the origin is

$$y = -x^2 - \mu x + \dots$$

and that the dynamics on the center manifold is

$$\dot{x} = \mu x - x^2 + O(x^3), \quad \dot{\mu} = 0.$$

(c) Construct phase portraits for the $x$-dynamics on the center manifold for the three cases $\mu > 0$, $\mu = 0$ and $\mu < 0$. Sketch the correponding bifurcation diagram.

6. The simple pendulum consists of a point particle of mass $m$ suspended from a fixed point by a massless rod of length $L$, which is allowed to swing in a vertical plane. If friction is ignored then the equation of motion is

$$\ddot{x} + \omega^2 \sin x = 0, \quad \omega^2 = \frac{g}{L},$$

where $x$ is the angle of inclination of the rod with respect to the downward vertical and $g$ is the gravitational constant.

(a) Using conservation of energy, show that the angular velocity of the pendulum satisfies

$$\dot{x} = \pm\sqrt{2}(C + \omega^2 \cos x)^{1/2},$$

where $C$ is an arbitrary constant. Express $C$ in terms of the total energy of the system.

(b) Sketch the phase diagram of the pendulum equation in the plane $(x, \dot{x})$ for $-3\pi \leq x \leq 3\pi$. Illustrate the one-parameter family of curves given by part (a) for different values of $C$. Indicate the fixed points of the system and the separatrices. Give a physical interpretation of the underlying trajectories in the two distinct dynamical regimes $|C| < \omega^2$ and $|C| > \omega^2$.

(c) Show that in the regime $|C| < \omega^2$, the period of oscillations is

$$T = 4\sqrt{\frac{L}{g}}K(\sin x_0/2),$$

where $\dot{x} = 0$ when $x = x_0$ and $K$ is the complete elliptic integral of the first kind, which is defined by

$$K(\alpha) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \alpha^2 \sin^2 u}} du.$$

HINT: Derive an integral expression for $T$ and then perform a change of variables

$$\sin u = \frac{\sin(x/2)}{\sin(x_0/2)}.$$