University of Utah, Department of Mathematics
Algebra 1 Qualifying Exam
August 2020

There are five problems on this exam. You may attempt as many problems as you wish; two correct solutions count as a *pass*, and three correct as a *high pass*. Show all your work, and provide reasonable justification for your answers.

1. Let R be a commutative Noetherian ring, and let $\varphi\colon R \longrightarrow R$ be a surjective ring homomorphism. Is φ necessarily an isomorphism?

2. Prove or disprove: each element of finite order in the general linear group $\mathrm{GL}_n(\mathbb{C})$ is diagonalizable.

3. Suppose M is a 3×3 matrix, with entries from the field of real numbers, such that

$$M^8 = I \quad \text{and} \quad M^4 \neq I.$$

   Determine the possibilities for the minimal polynomial of M.

4. Let I be the ideal (x,y) in the polynomial ring $R := \mathbb{C}[x,y]$, and let $\Lambda^2(I)$ denote the second exterior power of I in the category of R-modules. Prove or disprove: $\Lambda^2(I)$ is zero.

5. Determine all maximal ideals of the ring $\mathbb{C}[x,y]/(x^3 - x^2y,\ xy^2 + xy + x + 1)$.