UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Real Analysis
Aug 19, 2020.

Instructions. Answer as many questions as you can. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a low pass you need to solve *completely* at least two problems and score at least 25 points.

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1. Let $(X, \mathcal{M}, \mu)$ be a measure space and $E_n \in \mathcal{M}$ for $n = 1, 2, \cdots$. Assume that

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty$$

Prove that the set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

satisfies $\mu(E) = 0$.

2. (a) In the construction of the Lebesgue measure m on $\mathbb{R}$ a key step is the definition of the *outer measure* $m^*(E)$ of any subset $E \subset \mathbb{R}$. Define $m^*(E)$.

   (b) Suppose $E \subset \mathbb{R}$ has $m^*(E) > 0$. Show that for every $\alpha \in (0, 1)$ there is a finite interval $I \subset \mathbb{R}$ such that

$$m^*(E \cap I) > \alpha \; m(I)$$

3. Let $f_n \in L^2([0, 1], m)$ be a sequence of functions with $||f_n||_2 \leq 1$ for all n, where m is Lebesgue measure and $||f||_p$ is the L^p-norm. Assume that $f_n(x) \to 0$ for almost every x.

   (a) State Egoroff's theorem.

   (b) Show that $||f_n||_1 \to 0$.

   (c) Give an example showing that $||f_n||_2$ may not converge to 0.

4. Suppose $f \in L^p([0, \infty), m)$ for some $p \in [1, \infty]$, where m is Lebesgue measure. Compute

$$\lim_{n\to\infty} \int_0^{\infty} f(x) e^{-nx} dx$$

5. Let ℓ^∞ be the Banach space of bounded sequences $(x_i)_{i=1}^\infty$ of real numbers with the sup norm. Let $s \subset \ell^\infty$ be the subspace consisting of convergent sequences. Let $L : s \to \mathbb{R}$ be the functional

$$L((x_i)) = \lim_{i \to \infty} x_i$$

   (a) Prove that L extends to a bounded functional on ℓ^∞. If you use a named theorem, state precisely the version you are using.

   (b) Prove that ℓ^1 is not the dual of ℓ^∞, i.e. show that there is a bounded functional $F : \ell^\infty \to \mathbb{R}$ that is not equal to $(x_i) \mapsto \sum_{i=1}^\infty x_i y_i$ for any choice of $(y_i) \in \ell^1$.

6. Let $\mathcal{H}$ be a Hilbert space and $x_n \in \mathcal{H}$ a sequence such that $||x_n|| = 1$ for all n. Suppose that

$$\lim_{n,m \to \infty} ||x_n + x_m|| = 2$$

Prove that there exists $x \in \mathcal{H}$ such that

$$\lim_{n \to \infty} ||x_n - x|| = 0$$