Instructions. Answer as many questions as you can. For a high pass you need to solve completely at least three problems and score at least 30 points. For a low pass you need to solve completely at least two problems and score at least 25 points.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and $E_n \in \mathcal{M}$ for $n = 1, 2, \ldots$. Assume that
\[
\sum_{n=1}^{\infty} \mu(E_n) < \infty
\]
Prove that the set
\[
E = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n
\]
satisfies $\mu(E) = 0$.

2. (a) In the construction of the Lebesgue measure $m$ on $\mathbb{R}$ a key step is the definition of the outer measure $m^*(E)$ of any subset $E \subset \mathbb{R}$. Define $m^*(E)$.
(b) Suppose $E \subset \mathbb{R}$ has $m^*(E) > 0$. Show that for every $\alpha \in (0, 1)$ there is a finite interval $I \subset \mathbb{R}$ such that
\[
m^*(E \cap I) > \alpha m(I)
\]

3. Let $f_n \in L^2([0, 1], m)$ be a sequence of functions with $\|f_n\|_2 \leq 1$ for all $n$, where $m$ is Lebesgue measure and $\|f\|_p$ is the $L^p$-norm. Assume that $f_n(x) \to 0$ for almost every $x$.
(a) State Egoroff’s theorem.
(b) Show that $\|f_n\|_1 \to 0$.
(c) Give an example showing that $\|f_n\|_2$ may not converge to 0.

4. Suppose $f \in L^p([0, \infty), m)$ for some $p \in [1, \infty]$, where $m$ is Lebesgue measure. Compute
\[
\lim_{n \to \infty} \int_{0}^{\infty} f(x)e^{-nx} \, dx
\]
5. Let $\ell^\infty$ be the Banach space of bounded sequences $(x_i)_{i=1}^\infty$ of real numbers with the sup norm. Let $s \subset \ell^\infty$ be the subspace consisting of convergent sequences. Let $L : s \to \mathbb{R}$ be the functional

$$L((x_i)) = \lim_{i \to \infty} x_i$$

(a) Prove that $L$ extends to a bounded functional on $\ell^\infty$. If you use a named theorem, state precisely the version you are using.

(b) Prove that $\ell^1$ is not the dual of $\ell^\infty$, i.e. show that there is a bounded functional $F : \ell^\infty \to \mathbb{R}$ that is not equal to $(x_i) \mapsto \sum_{i=1}^\infty x_i y_i$ for any choice of $(y_i) \in \ell^1$.

6. Let $\mathcal{H}$ be a Hilbert space and $x_n \in \mathcal{H}$ a sequence such that $\|x_n\| = 1$ for all $n$. Suppose that

$$\lim_{n,m \to \infty} \|x_n + x_m\| = 2$$

Prove that there exists $x \in \mathcal{H}$ such that

$$\lim_{n \to \infty} \|x_n - x\| = 0$$