Alg. Top. Qual Exam

Name ______________________________

ID # u______________________________

Instructions:
Answer at most 5 of the problems below. Each problem is worth 10 points. If you answer more than 5 problems, let me know which 5 you would like me to grade. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a low pass you need to solve *completely* at least two problems and score at least 25 points.

1. Let a and b be the two generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove this covering space is indeed the correct one.

2. Describe a CW structure on $X = \mathbb{R}P^9/\mathbb{R}P^4$. Then compute the cellular homology of X.

3. The Klein bottle K can be decomposed as the union of two Möbius bands A and B, glued together by a homeomorphism between their boundary circles. Use this decomposition to compute the homology groups $H_i(K)$.

4. Let X_n be the topological space obtained by attaching a disk D to the torus $T = S^1 \times S^1$ where the attaching map is a degree n map from ∂D to $S^1 \times \{p\}$ in T.
   (a) Calculate $\pi_1(X_n)$.
   (b) Calculate the homology and cohomology of X_n with $\mathbb{Z}$ coefficients.

5. (a) Use van Kampen's Theorem to compute $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2)$.
   (b) Give the universal cover of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ (be sure to describe the covering map carefully).

6. Let Σ_g be the closed, orientable, genus-g surface. Show that $\pi_1(\Sigma_2)$ contains $\pi_1(\Sigma_4)$ as a normal subgroup.

7. Prove that the map $p_* : \pi_1(\widetilde{X}, \tilde{x}) \to \pi_1(X, x)$ induced by a covering map $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ is injective.

Positive script for test anxiety

If you are feeling anxious, reading the following might help:

Relax and take three deep breaths. Do not panic. You have studied hard to prepare for this exam and you know the material. Focus on one item at a time, not on the whole test.
It's OK if an answer does not come to you right now, you can go on and try later.

Question 1

Let a and b be the two generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove this covering space is indeed the correct one.

Question 2

Describe a CW structure on $X = \mathbb{R}P^9/\mathbb{R}P^4$. Then compute the cellular homology of X.

Question 3

The Klein bottle K can be decomposed as the union of two Möbius bands A and B, glued together by a homeomorphism between their boundary circles. Use this decomposition to compute the homology groups $H_i(K)$.

Question 4

Let X_n be the topological space obtained by attaching a disk D to the torus $T = S^1 \times S^1$ where the attaching map is a degree n map from ∂D to $S^1 \times \{p\}$ in T.

(a) Calculate $\pi_1(X_n)$.

(b) Calculate the homology and cohomology of X_n with $\mathbb{Z}$ coefficients.

Question 5

(a) Use van Kampen's Theorem to compute $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2)$.

(b) Give the universal cover of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ (be sure to describe the covering map carefully).

Question 6

Let Σ_g be the closed, orientable, genus-g surface. Show that $\pi_1(\Sigma_2)$ contains $\pi_1(\Sigma_4)$ as a normal subgroup.

Question 7

Prove that the map $p_* : \pi_1(\widetilde{X}, \tilde{x}) \to \pi_1(X, x)$ induced by a covering map $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ is injective.