DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN COMPLEX ANALYSIS
August 2021

Instructions: This exam has two sections, A and B. Three (3) completely correct problems in section A will be a high pass and any three (3) completely correct problems is a passing exam. Note that solutions for problems in section B will not be counted towards a high pass. Be sure to provide detailed proofs and all relevant definitions and statements of theorems cited.

Let $B(x,r) = \{z \in \mathbb{C} : |z - x| < r\}$.

A

1. Prove that if $(f, B(0,1))$ is a function element, $\{(g_t, U_t)\}_{t\in[0,1]}$ and $\{(h_t, V_t)\}_{t\in[0,1]}$ are analytic continuations of $(f, B(0,1))$ along the curve $\phi : [0,1] \to \mathbb{C}$ defined by $\phi(t) = 4t + it$ then $g_1 = h_1$ in a neighborhood of $4 + i$.

2. Without quoting Picard's theorems, show that if f is entire and not a polynomial then for every $\epsilon > 0$ there exists $p \in \mathbb{C}$ so that $|f(p) - p^2| < \epsilon$.

3. State and prove Rouché's theorem.

4. Show that if $D \subsetneq B(0,1)$ is simply connected and $0 \in D$ then there exists an injective, holomorphic function $f : D \to B(0,1)$ so that $f'(0) > 1$.

   -Note that this is part of the proof of the Riemann mapping theorem, so please do not use the Riemann mapping theorem in the proof.

5. Prove the following special case of Runge's approximation theorem: If
$$f : B(0,4) \setminus (B(0,.1) \cup B(1,.1)) \to \mathbb{C}$$
is holomorphic and $\epsilon > 0$ is given then there exists q, a rational function, so that $|f(z) - q(z)| < \epsilon$ for all $z \in \overline{B(0,2) \setminus (B(0,.2) \cup B(1,.2))}$.

6. Let $D \subset \mathbb{C}$ be a connected open set and $f_1, ... : D \to \mathbb{C}$ be a sequence of holomorphic functions that converges uniformly on compact sets to a function g. Show that g is holomorphic and $g'(z) = \lim_{j\to\infty} f_j'(z)$ for all $z \in D$.

B

1. Show that if $f, g : B(0,1) \setminus \{0\} \to \mathbb{C}$, are holomorphic, f has an essential singularity at 0 and g has a pole at 0 then $\frac{g}{f}$ has an essential singularity at 0.
2. Give an example of a domain D and a holomorphic function $f : D \to \mathbb{C}$, so that f is not given by a single Laurent series on D. (Please include a proof.)