UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Real Analysis
Aug 18, 2021.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

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1. Let $(X, \mathcal{M})$ be a measurable space, so that $\mathcal{M}$ is a σ-algebra of subsets of X. Prove that $\mathcal{M}$ is either finite or uncountable.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$ and let $E_1, \cdots, E_n$ be measurable subsets of X. Suppose that every $x \in X$ belongs to at least k of the sets E_i. Show that at least one of the sets E_i has measure $\mu(E_i) \geq \frac{k}{n}$.

3. Show that there does not exist a Lebesgue measurable subset $A \subset \mathbb{R}$ such that for every interval $(a, b) \subset \mathbb{R}$ we have
$$m(A \cap (a, b)) = \frac{b-a}{2}$$
where m denotes Lebesgue measure.

4. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and define a signed measure μ on the Lebesgue measurable subsets of $[0, 1]$ by
$$\mu(E) = \int_E f \, dm$$
where m is Lebesgue measure. Suppose that for all $n = 0, 1, 2, 3 \cdots$
$$\int_{[0,1]} x^n d\mu = 0$$
Prove that $\mu = 0$.

5. Let $f, g \in L^2(\mathbb{R})$. Show that
$$h(x) = \int_{\mathbb{R}} f(x - y)g(y) \, dy$$
is a well-defined real number for every $x \in \mathbb{R}$ and that $h : \mathbb{R} \to \mathbb{R}$ is continuous.

6. Prove or disprove:

   (i) ℓ^1 is separable.

   (ii) ℓ^∞ is separable.