Probability Qualifying Exam

August 2021

Instructions (Read before you begin)

- You may attempt all 6 problems in this exam. However, you can turn in solutions for **at most** 4 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is 10 points; a pass is 30 points or higher; a high pass is 36 points or higher.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Let X, Y, Z be random variables on some probability space. Recall that

$$\mathrm{P}(X \in A \,|\, Y, Z) = \mathrm{P}(X \in A \,|\, \sigma\{Y, Z\})$$

is the conditional expectation of the indicator function of the event $\{X \in A\}$, given the σ-algebra $\sigma\{Y, Z\}$ generated by Y and Z. We say that X and Y are independent, given Z, if for all Borel sets $A, B \subset \mathbb{R}$,

$$\mathrm{P}(X \in A, Y \in B \,|\, Z) = \mathrm{P}(X \in A \,|\, Z) \cdot \mathrm{P}(Y \in B \,|\, Z) \quad \text{P-almost surely.}$$

Show that this condition is equivalent to having

$$\mathrm{P}(X \in A \,|\, Y, Z) = \mathrm{P}(X \in A \,|\, Z) \quad \text{P-almost surely,} \quad \text{for all Borel sets } A.$$

2. Let Y_k be i.i.d., $Y_k \geqslant 0$, and $\mathrm{E}[Y_k] \leqslant 1$ for all k. Prove the existence of the almost sure limit $M_\infty = \lim_{n\to\infty} \prod_{i=1}^n Y_i$ and describe this random variable explicitly. Do not forget the degenerate cases.

3. Let $\{\mathcal{F}_n\}_{n\geqslant 0}$ be a filtration on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$, let $X \in L^1(\Omega, \mathcal{F}, \mathrm{P})$, and define $M_n = \mathrm{E}[X \,|\, \mathcal{F}_n]$. Let τ be a stopping time and recall the definition

$$\mathcal{F}_\tau = \big\{A \in \mathcal{F} : A \cap \{\tau = n\} \in \mathcal{F}_n \ \forall n \geqslant 0\big\}.$$

(a) Show that $\mathbf{1}\{\tau < \infty\} M_\tau$ is integrable.

(b) Show that $\mathrm{E}[X \,|\, \mathcal{F}_\tau] = \mathbf{1}\{\tau < \infty\} M_\tau + \mathbf{1}\{\tau = \infty\} X$. Do not forget the necessary measurability condition.

4. Answer the following questions.

   (a) State the definition of convergence in probability. Using only this definition, show that $X_n \to X$ in probability if and only if $\mathrm{E}[1 \wedge |X_n - X|] \to 0$. Do not appeal to any theorems that trivialize the problem.

   (b) State the definition of almost sure convergence. Prove that if $X_n \to X$ almost surely, then $X_n \to X$ in probability. Again, do not appeal to any theorems that trivialize the problem.

5. Let $\{X_n\}_{n \geqslant 1}$ be i.i.d. with mean 0 and variance 1. Let $\{a_n\}_{n \geqslant 1}$ be a sequence of nonzero real numbers and $A_n = \left(\sum_{k=1}^{n} a_k^2\right)^{1/2}$. Assume that

$$A_n^{-1}\left(\max_{1 \leqslant k \leqslant n} |a_k|\right) \to 0 \quad \text{as } n \to \infty.$$

   Find (and prove) a distributional limit for $A_n^{-1} \sum_{k=1}^{n} a_k X_k$ as $n \to \infty$.

6. Let $\{X_n\}_{n \geqslant 1}$ be i.i.d. Normal random variables with mean 0 and variance 1. Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geqslant 1$. Fix $t \in \mathbb{R}$ and for $n \geqslant 0$ let $M_n = e^{tS_n - t^2 n/2}$.

   (a) Prove that M_n is a mean one martingale relative to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

   (b) Prove that if $\max_{j \leqslant n} S_j \geqslant nt$, then $\max_{j \leqslant n} M_j \geqslant e^{t^2 n/2}$.

   (c) Conclude that $\mathrm{P}\left\{\max_{j \leqslant n} S_j \geqslant nt\right\} \leqslant e^{-t^2 n/2}$ and then that for any $c > \theta > 1$ we have with probability one: $\exists k_0 \geqslant 1$ such that for $k \geqslant k_0$

$$\frac{\max_{j \leqslant \theta^k} S_j}{\sqrt{2c\theta^{k-1} \log\log \theta^{k-1}}} \leqslant 1.$$

   (d) Deduce that almost surely

$$\overline{\lim_{n \to \infty}} \frac{S_n}{\sqrt{2n \log\log n}} \leqslant 1.$$