Probability Qualifying Exam

August 2021

Instructions (Read before you begin)
- You may attempt all 6 problems in this exam. However, you can turn in solutions for at most 4 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is 10 points; a pass is 30 points or higher; a high pass is 36 points or higher.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:
1. Let \( X, Y, Z \) be random variables on some probability space. Recall that
   \[
P(X \in A \mid Y, Z) = P(X \in A \mid \sigma\{Y, Z\})
   \]
   is the conditional expectation of the indicator function of the event \( \{X \in A\} \), given the \( \sigma \)-algebra \( \sigma\{Y, Z\} \) generated by \( Y \) and \( Z \). We say that \( X \) and \( Y \) are independent, given \( Z \), if for all Borel sets \( A, B \subset \mathbb{R} \),
   \[
P(X \in A, Y \in B \mid Z) = P(X \in A \mid Z) \cdot P(Y \in B \mid Z) \quad P\text{-almost surely.}
   \]
   Show that this condition is equivalent to having
   \[
P(X \in A \mid Y, Z) = P(X \in A \mid Z) \quad P\text{-almost surely, \ for all Borel sets } A.
   \]
2. Let \( Y_k \) be i.i.d., \( Y_k \geq 0 \), and \( E[Y_k] \leq 1 \) for all \( k \). Prove the existence of the almost sure limit \( M_\infty = \lim_{n \to \infty} \prod_{i=1}^{n} Y_i \) and describe this random variable explicitly. Do not forget the degenerate cases.
3. Let \( \{\mathcal{F}_n\}_{n \geq 0} \) be a filtration on the probability space \( (\Omega, \mathcal{F}, P) \), let \( X \in L^1(\Omega, \mathcal{F}, P) \), and define \( M_n = E[X \mid \mathcal{F}_n] \). Let \( \tau \) be a stopping time and recall the definition
   \[
   \mathcal{F}_\tau = \{ A \in \mathcal{F} : A \cap \{\tau = n\} \in \mathcal{F}_n \ \forall n \geq 0 \}.
   \]
   (a) Show that \( 1\{\tau < \infty\}M_\tau \) is integrable.
   (b) Show that \( E[X \mid \mathcal{F}_\tau] = 1\{\tau < \infty\}M_\tau + 1\{\tau = \infty\}X \). Do not forget the necessary measurability condition.
4. Answer the following questions.

(a) State the definition of convergence in probability. Using only this definition, show that $X_n \to X$ in probability if and only if $E[1 \wedge |X_n - X|] \to 0$. Do not appeal to any theorems that trivialize the problem.

(b) State the definition of almost sure convergence. Prove that if $X_n \to X$ almost surely, then $X_n \to X$ in probability. Again, do not appeal to any theorems that trivialize the problem.

5. Let $\{X_n\}_{n \geq 1}$ be i.i.d. with mean 0 and variance 1. Let $\{a_n\}_{n \geq 1}$ be a sequence of nonzero real numbers and $A_n = \left(\sum_{k=1}^{n} a_k^2\right)^{1/2}$. Assume that

$$A_n^{-1} \max_{1 \leq k \leq n} |a_k| \to 0 \quad \text{as } n \to \infty.$$ 

Find (and prove) a distributional limit for $A_n^{-1} \sum_{k=1}^{n} a_k X_k$ as $n \to \infty$.

6. Let $\{X_n\}_{n \geq 1}$ be i.i.d. Normal random variables with mean 0 and variance 1. Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Fix $t \in \mathbb{R}$ and for $n \geq 0$ let $M_n = e^{tS_n - t^2n/2}$.

(a) Prove that $M_n$ is a mean one martingale relative to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

(b) Prove that if $\max_{j \leq n} S_j \geq nt$, then $\max_{j \leq n} M_j \geq e^{t^2 n/2}$.

(c) Conclude that $P\{\max_{j \leq n} S_j \geq nt\} \leq e^{-t^2 n/2}$ and then that for any $c > \theta > 1$ we have with probability one: $\exists k_0 \geq 1$ such that for $k \geq k_0$

$$\frac{\max_{j \leq \theta^k} S_j}{\sqrt{2c\theta^{k-1}\log \log \theta^{k-1}}} \leq 1.$$ 

(d) Deduce that almost surely

$$\lim_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} \leq 1.$$