Read the following instructions before you begin:

- You may attempt all of 10 problems in this exam. On the outside of your exam booklet, indicate which problems you are turning in and want graded.
- Each problem is worth 10 points; 60 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 5080–5090/6010 texts, then you need to carefully state that result.

Exam problems begin here:

1. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(t, \theta) = \begin{cases} 
0, & \text{if } t < 0 \\
\frac{1}{\theta} e^{-t/\theta}, & \text{if } t \geq 0.
\end{cases}$$

Let $\alpha > 0$ and

$$\tau = \frac{1}{\theta^\alpha}.$$ 

(a) How big $n$ should be (as a function of $\alpha$) so that $\tau$ has uniformly minimum variance unbiased estimator?

(b) Find the uniformly minimum variance unbiased estimator for $\tau$ (you need to justify that your estimator has the required property).

2. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(t, \theta) = \begin{cases} 
0, & \text{if } t \notin [-\theta, \theta] \\
\frac{5t^4}{2\theta^5}, & \text{if } -\theta \leq t \leq \theta.
\end{cases}$$

Find the maximum likelihood estimator for $\theta$ and compute its bias.
3. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(t, \theta) = \begin{cases} 
0, & \text{if } t \not\in [-\theta, \theta] \\
\dfrac{10t^9}{2\theta^9}, & \text{if } -\theta \leq t \leq \theta.
\end{cases}$$

Find a moment estimator for $\theta$.

4. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(t; \theta) = \frac{1}{2} e^{-|t-\theta|}, \quad -\infty < t < \infty.$$ 

The parameter $\theta$ could be any real number. Find all maximum likelihood estimators for $\theta$.

5. Let $X_1, X_2, \ldots, X_n$ be iid with the same pdf as in question 5. Find the form of the rejection region coming from the generalized likelihood ratio test for $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$.

6. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables that have the uniform distribution on $[0,1]$. Let $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ be the order statistics. Let $k \geq 1$ be a fixed integer. Show that $Y_n = nX_{k,n}$ converges in distribution and compute the limiting distribution.

7. Let $X$ and $Y$ be independent identically distributed random variables with density function

$$f(t, \theta) = \begin{cases} 
0, & \text{if } t < 0 \\
\frac{1}{\theta} e^{-t/\theta}, & \text{if } t \geq 0.
\end{cases}$$

Compute the density function of

$$Z = \frac{X}{X+Y}.$$ 

8. Let $X_1, \ldots, X_n \sim N(\mu_1, 1)$ and $Y_1, \ldots, Y_m \sim N(\mu_2, 1)$, where all random variables are independent. Derive the likelihood ratio test for $H_0 : \mu_1^2 = \mu_2^2$ vs. $H_a : \mu_1^2 \neq \mu_2^2$.

What is the distribution of the test statistic?

9. Suppose a box contains 6 marbles in total. Suppose $\theta$ of them are white and $6 - \theta$ of them are black. Test $H_0 : \theta = 2$ against $H_a : \theta \neq 2$ as follows: draw two marbles without replacement and reject $H_0$ if both marbles are the same color, otherwise do not reject. What is the probability of Type II error for each of the alternatives $\theta = 1$ and $\theta = 3$?

10. Let $X_1, \ldots, X_n$ be iid random variables. The density function of $X_i$ is

$$g(t; \theta_i) = \theta_i t^{-\theta_i - 1} 1\{t \geq 1\}.$$ 

Here $\theta_i > 0$ is an unknown parameter. We wish to test

$$H_0 : \theta_1 = \theta_2 = \ldots = \theta_n$$

against the alternative that $H_0$ is not true. Find a test using the generalized likelihood ratio.