Statistics Qualifying Exam

January, 2010

You need to correctly solve 8 of the following problems to guarantee a “pass”.

1. Let \( X_1, X_2, \ldots, X_n \) be independent, identically distributed random variables with distribution function

\[
F(t) = \begin{cases} 
0, & \text{if } -\infty < t < 0 \\
t^3, & \text{if } 0 \leq t \leq 1 \\
1, & \text{if } t > 0.
\end{cases}
\]

Show that \( Y_n = n^{1/3}X_{1,n} \) converges in distribution, where \( X_{1,n} = \min\{X_1, X_2, \ldots, X_n\} \).

2. Let \( X_1, X_2, \ldots, X_n \) be independent identically distributed random variables with density function

\[
h(t, \theta) = \begin{cases} 
0, & \text{if } t \notin [-\theta, \theta] \\
\frac{5}{2} \theta^{-5} t^4, & \text{if } -\theta \leq t \leq \theta.
\end{cases}
\]

Find a moment estimator for \( \theta \).

3. Let \( X_1, \ldots, X_n \) be independent, identically distributed random variables with density function

\[
h(t, \theta) = \begin{cases} 
0, & \text{if } -\infty < t < 0 \\
\theta(t + 1)^{-\theta-1}, & \text{if } 0 \leq t < \infty,
\end{cases}
\]

\( \theta > 0 \).

(a) Find the maximum likelihood estimator for \( \theta \).

(b) Is the estimator unbiased?

(c) Find the asymptotic variance of the maximum likelihood estimator for \( \theta \).
4. Let \( X_1, X_2, \ldots, X_n \) be independent identically distributed random variables with density function

\[
h(t, \theta) = \begin{cases} 
0, & \text{if } t \not\in [0, \theta] \\
\frac{2\theta^{-2}t}{2} & \text{if } 0 \leq t \leq \theta.
\end{cases}
\]

Find the uniformly minimum variance unbiased estimator for \( \theta \). Explain your answer. You need to prove directly that the sufficient statistic is also complete in this case.

5. Let \( X \) and \( Y \) be two independent random variables with density functions

\[
f(t) = \begin{cases} 
0, & \text{if } t \not\in [0, 4] \\
\frac{1}{4}, & \text{if } 0 \leq t \leq 4
\end{cases}
\]

and

\[
h(t) = \begin{cases} 
0, & \text{if } -\infty < t < 0 \\
2e^{-2t} & \text{if } 0 \leq t < \infty.
\end{cases}
\]

Compute the density of \( X - Y \).

6. Let \( X_1 \) and \( X_2 \) be independent random variables. The density function of \( X_1 \) is

\[
f(t) = \begin{cases} 
0, & \text{if } t \not\in [0, 1] \\
\frac{e^t}{e - 1} & \text{if } 0 \leq t \leq 1.
\end{cases}
\]

The distribution of \( X_2 \) is

\[
P\{X_2 = 1\} = p \quad \text{and} \quad P\{X_2 = -1\} = q, \quad p + q = 1.
\]

Compute the moment generating function of \( X_1X_2 \).

7. Let \( X_1, \ldots, X_n \) be an i.i.d. sample from a \( \text{UNIF}(0, \theta) \) distribution, where \( \theta > 0 \) is unknown. Find a 95% confidence interval for \( \theta \).

8. Let \( X_1, \ldots, X_n \) denote an independent sample from an exponential distribution with \( \text{[unknown]} \) mean \( \theta > 0 \). What does the Neyman–Pearson lemma say about \( H_0 : \theta = 1 \) versus \( H_a : \theta = 2 \)? Explain carefully, and identify explicitly the rejection region.

9. Let \( m \) denote the median distance [in 1000 miles] required for a certain brand of automobile tires to wear out. Test to see whether or not \( m \leq 29 \), based on the following random sample:

\[
23, 20, 26, 25, 48, 26, 25, 24, 15, 20
\]
10. Derive, using only first principles, the least-squares estimators of the slope and the intercept of a linear regression problem. What can you say about the optimality properties of those estimators?

11. The following data are times (in hours) between failures of air conditioning equipment in a particular airplane:

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Assume that the data are observed values of an i.i.d. random sample from an exponential distribution, \( X_i \sim \text{EXP}(\theta) \). Test \( H_0 : \theta = 125 \) versus \( H_a : \theta \neq 125 \). (A chi-square table is provided.)

12. A sample of 400 people was asked their degree of support of a balanced budget and their degree of support of public education, with the following results:

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<th>Undecided</th>
<th>Weak</th>
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Test the hypothesis of independence at \( \alpha = 0.05 \). (A chi-square table is provided.)
### TABLE 4

100 × γth Percentiles $\chi^2_γ(v)$ of the chi-square distribution with $v$ degrees of freedom

\[ y = \int_0^\infty \frac{y^{v-1}e^{-y/2}}{\Gamma(v/2)} \, dy \]

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<th>0.005</th>
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For large $v$, $\chi^2_γ(v) \approx ν(1 - (2/9ν) + z^2/(2ν))^2$. 

4