You need at least 50 points to pass.

1. Let $U_1, U_2, \ldots, U_n$ be independent random variables uniform on $[0, 1]$ and let $U_{1,n} \leq U_{2,n} \leq \ldots \leq U_{n,n}$ be the corresponding order statistics.
   a. Compute the asymptotic distribution of $nU_{2,n}$. (5 points)
   b. Show that $nU_{1,n}$ and $n(1 - U_{n,n})$ are asymptotically independent. (5 points)

2. Let $y_i = \alpha x_i + \epsilon_i$, $1 \leq i \leq n$, where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent identically distributed normal $N(0, \sigma^2)$ random variables.
   a. Determine the maximum likelihood estimators for $\alpha$ and $\sigma^2$. (5 points)
   b. Compute the distribution of the maximum likelihood estimator for $\alpha$. (5 points)
   c. Compute the distribution of the maximum likelihood estimator for $\sigma^2$. (5 points)

3. Let $X$ and $Y$ be two independent random variables with distribution functions $F$ and $G$ and density functions $f$ and $g$.
   a. Determine the distribution function of $XY$. (5 points)
   b. Does $XY$ always have a density function? (5 points)
   c. Assuming that $XY$ has a density function, compute it. (5 points)

4. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with density function

   \[ f(t; \theta) = \begin{cases} 
   0, & \text{if } t < \theta \\
   e^{-(t-\theta)}, & \text{if } t > \theta.
   \end{cases} \]

   a. Find a moment estimator for $\theta$. (5 points)
   b. Find the maximum likelihood estimator for $\theta$. (5 points)
   c. Determine the asymptotic efficiency between the two estimators. (5 points)

5. The number of customers entering a store on a given day is a Poisson random variable with parameter $\theta$. The money spent by a customer is uniformly distributed on $[0, \eta]$. The number of customers and the money spent in the store are independent. We observe $Z_1, Z_2, \ldots, Z_n$, the total spending (the money collected by the store) on $n$ days. We can assume that $Z_1, Z_2, \ldots, Z_n$ are independent.
   a. Provide estimators for $\theta$ and $\eta$ using the method of moments. (10 points)
   b. Provide estimators for $\theta$ and $\eta$ using the likelihood method. (10 points)

6. Let $\Phi$ and $\phi$ denote the standard normal distribution and density functions.
   (a) Prove that for all $x > 0$

   \[ 1 - \Phi(x) \leq \frac{1}{x} \phi(x). \]
(5 points)
Let \( X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n} \) denote the order statistics from a sample of \( n \) independent identically distributed standard normal random variables.

(b) Show that \( X_{n,n} \) converges in probability to infinity. (5 points)

(c) Show that for all \( \epsilon > 0 \)

\[
\lim_{n \to \infty} P\{X_{n,n} \leq \sqrt{(2 + \epsilon) \log n}\} = 1.
\]

(5 points)

7. Let \( X_1, \ldots, X_n \) be independent identically distributed Poisson random variables with parameter \( \theta_1 \). Let \( Y_1, \ldots, Y_m \) be independent identically distributed Poisson random variables with parameter \( \theta_2 \). We assume that the two samples are independent. We wish to test \( H_0 : \theta_1 = \theta_2 \) against the alternative that \( H_0 \) is not true.

a. Derive the likelihood ratio test. (5 points)

b. Provide a large sample approximation for the rejection region. (2 points)

8. Let \( X_1, \ldots, X_n \) be independent identically distributed random variables with distribution function \( F \). We assume that \( F \) is strictly increasing and continuous on its support. Let \( x_{1/2} \) denote the median.

a. Find a \( 1 - \alpha \) confidence interval for \( x_{1/2} \). (5 points)

b. Provide a large sample approximation for the confidence interval. (5 points)

9. Let \( X_1, \ldots, X_N \) be independent random variables. The distribution of \( X_i \) is binomial(\( n_i, p_i \)). We wish to test \( H_0 : p_1 = p_2 = \ldots = p_N \) against the alternative that \( H_0 \) is not true.

a. Derive the likelihood ratio test. (5 points)

b. Provide a large sample approximation for the rejection region. (2 points)