A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

Notation: $\lambda$ denotes Lebesgue measure on $\mathbb{R}$ (or a subset of it).
Let $\lambda$ denote Lebesgue measure on $\mathbb{R}$ (or a subset of it). Let $\mathcal{H}$ be (an infinite dimensional) Hilbert space.

1. Prove or disprove: If $(X, \mathcal{M}, \mu)$ is a measure space, $A_1, \ldots$ are measurable and $\sum \mu(A_i)^2 < \infty$ then $\mu(\cap_{m=1}^{\infty} \cup_{i=m}^{\infty} A_i) = 0$.

2. Prove or give a counterexample: $B_1, B_2$ are Banach spaces and $A : B_1 \to B_2$ is continuous, linear, injective and $A(B_1)$ is dense then $A$ is a bijection.

3. Show that if $v_i \in \mathcal{H}$ and $v_i$ converges weakly to $u$ then $\liminf_{i \to \infty} \|v_i\| \geq \|u\|$ and that equality occurs only where there exists a subsequence $v_{n_j}$ so that $v_{n_j}$ converges to $u$ in norm.

4. Prove that the convolution of two $L^1(\lambda)$ functions is in $L^1(\lambda)$.

5. This question considers different notions of convergence for $\ell^1(\mathbb{N})$:
   (a) Does pointwise convergence imply $\| \cdot \|_1$ convergence? Vice versa?
   (b) Does pointwise convergence imply convergence in measure? Vice Versa?
B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

6. Let \( f \) be a function holomorphic for \( z \) in the annulus \( r < z < R \). Show that if for some \( \rho \) with \( r < \rho < R \),
\[
\int_{|z|=\rho} z^n f(z) \, dz = 0
\]
for all nonnegative \( n \) then \( f \) extends to a holomorphic function on the disk \( |z| < R \).

7. Let \( f \) be an entire function such that for each \( z \in \mathbb{C} \), \( \Re f(z) \cdot \Im f(z) \neq 1 \). Prove that \( f \) is a constant.

8. Using methods of complex analysis, evaluate
\[
\int_0^\infty \frac{x^2 \, dx}{x^4 + x^2 + 1}
\]

9. Let \( f : \mathbb{C} \to \mathbb{C} \) be a function such that:
   (a) \( f \) is continuous;
   (b) \( f \) is a holomorphic at all points \( z \) not on the real axis;
   (c) if \( z \) is on the real axis then \( f(z) \) is real.
Show that \( f(\bar{z}) = \overline{f(z)} \) and that \( f \) is holomorphic on all of \( \mathbb{C} \).

10. Determine the number of zeroes of the polynomial
\[
z^5 + z^3/3 + z^2/4 + z/3
\]
in the region \( 1/2 < |z| < 1 \).