Instructions: Do seven problems and list on the front of your blue book the seven problems to be graded. Do at least three problems from each part. Two correct solutions from each section will represent a passing exam.

Part A:
Let $\lambda$ denote Lebesgue measure on $\mathbb{R}$. Let $F$ denote the Fourier transform.

Problem 1. If $f \in L^1(\lambda)$ then $\lim_{c \to 1} \int |f(cx) - f(x)|d\lambda = 0$.

Problem 2. If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz and $A$ is $\lambda$-measurable then $f(A)$ is measurable. Recall that a function is Lipschitz if there exists $C$ so that $d(fx, fy) \leq Cd(x, y)$ for all $x, y \in \mathbb{R}$.

Problem 3. Show that $f \in L^1$ then $\lim_{x \to \infty} F(f)(x) = 0$.

Problem 4. Show that $\{\vec{v} \in \ell^p : |v_i| \leq 2^{-i}\}$ is compact for all $p \geq 1$.

Problem 5. Prove that if $\mu$ is a non-atomic Borel probability measure on $\mathbb{R}$ then $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \mu((\infty, t))$ is continuous.
Part B:

Problem 1. Evaluate the integral by the method of residue:

\[
\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} \, dx,
\]
for \(a \in \mathbb{R}\).

Problem 2. Let \(f(z)\) be analytic in a deleted neighborhood of \(z = a\) and
\(\lim_{z \to a} (z - a)f(z) = 0\). Does it mean that \(f(z)\) has removable singularity
at \(a\)? If so, prove it. If not, give a counterexample.

Problem 3. If \(f(z)\) is analytic in \(|z| \leq 1\) and satisfies \(|f| = 1\) on \(|z| = 1\),
show that \(f(z)\) is rational.

Problem 4. Prove that in any region \(\Omega\) the family of analytic functions
whose values lie on an open half plane is normal.

Problem 5. Let \(\wp(z)\) be the Weierstrass \(\wp\) function. Show that any even
elliptic function \(f(z)\) with periods \(\omega_1, \omega_2\) can be expressed in the form
\[
(C) \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}
\]
for some Constant, \(a_k, b_k \in \mathbb{C}\), provided that 0 is neither a zero or a pole.
What is the corresponding form if \(f(z)\) either vanishes or has poles at \(z = 0\)?