Department of Mathematics
University of Utah
Real and Complex Analysis Preliminary Examination

Instructions: Do seven problems and list on the front of your blue book the seven problems to be graded. Do at least three problems from each part. Two correct solutions from each section will represent a passing exam.

Part A:

Let λ denote Lebesgue measure on $\mathbb{R}$. If X is a set let $\mathcal{P}(X)$ denote the power set of X.

Problem 1. For any subset A of $\mathbb{R}$ we define its Lebesgue outer measure

$$\lambda^*(A) = \inf \left\{ \sum_{i=0}^{\infty} |b_i - a_i| \,\middle|\, A \subset \bigcup_{i=0}^{\infty} (a_i, b_i) \right\}.$$

Recall that given an outer measure, $\phi^* : \mathcal{P}(X) \to [0, \infty]$, we say a set A is ϕ^*-measurable if for all $Y \in \mathcal{P}(X)$ we have $\phi^*(Y) = \phi^*(Y \cap A) + \phi^*(Y \setminus A)$. Show that if $\lambda^*(\partial E) = 0$ then E is λ^*-measurable.

Problem 2. A family of complex-valued functions $\mathcal{S}$ on $[0,1]$ is *uniformly integrable* if for all $\epsilon > 0$ there exists $\delta > 0$ so that $\lambda(A) < \delta$ implies $|\int_A f\, d\lambda| < \epsilon$ for all $f \in \mathcal{S}$.

Let $(f_i; i \in \mathbb{N})$ be uniformly integrable sequence. Assume that

$$\lim_{i\to\infty} f_i(x) = g(x) \text{ for all } x \in [0,1].$$

Prove that

$$\lim_{i\to\infty} \int_0^1 f_i\, d\lambda = \int_0^1 g\, d\lambda.$$

Problem 3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ be a Borel measurable function. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the map given by $g(x,y) = (2x + y^2, y)$ for $(x,y) \in \mathbb{R}^2$. If $f \in L^1(\lambda^2)$ what is

$$\frac{\int_{\mathbb{R}^2} (f \circ g)\, d\lambda^2}{\int_{\mathbb{R}^2} f\, d\lambda^2}?$$

Problem 4. Let $\mathcal{H}$ be a Hilbert space. Prove every orthonormal set is contained in an orthonormal basis.

Problem 5. Consider $\ell^4(\mathbb{N})$ equipped with topology induced by the norm $\|\cdot\|_4$. Show that there exists an open set in this topology that is not open in the weak topology on $\ell^4(\mathbb{N})$.

Part B:

Problem 1. What are the possible ranges of holomorphic functions from $\mathbb{C} \setminus \{\pi\}$ to $\mathbb{C}$?

Problem 2. Let $\mathbb{D}$ be the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$. Let f be a holomorphic function on $\mathbb{D} \setminus \{0\}$ such that

$$|f(z)| \le \log\left(\frac{1}{|z|}\right) \quad \text{for all } z \in \mathbb{D} \setminus \{0\}.$$

Show that $f = 0$.

Problem 3. Let f be a holomorphic function on the unit disk $\mathbb{D}$. Then there exists $n \in \mathbb{N}$ so that

$$f\left(\frac{1}{n}\right) \neq \frac{1}{n+2}.$$

Problem 4. Let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function (where U is an open set). Let $a \in U$. A standard result in complex analysis is that if $f'(a) \neq 0$ then there is an open neighborhood $V \subset U$ of a on which f is injective and the image $f(V)$ is a neighborhood of $f(a)$. Prove this result!

Problem 5. Using the residue theorem find the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 20}\, dx.$$