Instructions: Do seven problems, at least three from part A and three from part B. List the problems you have done on the front of your blue book.

Part A.

1. Let \((X, \mu)\) be a measure space and \(f \geq 0\) an integrable function on \(X\). Prove that for \(\epsilon > 0\), there exists \(\delta > 0\) such that whenever \(\mu(A) < \delta\), then \(\int_A f < \epsilon\).

2. Let \(K\) be a continuous function on the square \([0, 1] \times [0, 1]\), and let \([0, 1]\) be endowed with the Lebesgue measure. For \(f \in L^2([0, 1])\) define \(Tf(x) = \int_0^1 K(x, y)f(y)\, dy\). Show that \(Tf \in L^2([0, 1])\) and that \(T : L^2 \to L^2\) is a bounded operator with \(\|T\| \leq \|K\|_\infty\).

3. Let \(H\) be a Hilbert space, and let \(T : H \to H\) be a self adjoint linear operator, with finite dimensional range \(V = T(H)\). Prove that \(T\) is a compact operator, and that \(V = K^\perp\), where \(K\) is the kernel of \(T\).

4. Let \(f \in L^1([-\pi, \pi])\), and let \(c_n\) be the \(n\)th Fourier coefficient of \(f\):

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \quad n \in \mathbb{Z}
\]

Prove that \(c_n \to 0\) as \(|n| \to \infty\).

5. Let \(\ell^\infty\) denote the Banach space of bounded real sequences with norm \(\|x\|_\infty = \sup_i |x_i|\), and \(\ell^1\) the Banach space of real sequences with norm \(\|x\|_1 = \sum_i |x_i| < \infty\). For \(x \in \ell^\infty\) and \(y \in \ell^1\) define \(T(x) = \sum_i x_i y_i\).

Prove that \(T : \ell^\infty \to \ell^{1*}\) is a bounded operator that is one to one and onto. (Here \(\ell^{1*}\) is the dual space of \(\ell^1\).)

Part B.

6. Let \(Q\) be a square in \(C\) and \(f : Q \to C\) a continuous map that is holomorphic on the interior of \(Q\). Also assume that if \(z \in \partial Q\) then \(|f(z)| = 1\). Show that \(f\) extends to a holomorphic map on a neighborhood of \(Q\).

7. Show that the function \(f(z) = \frac{\cos z}{z^2}\) is the complex derivative of a holomorphic function \(F\) on \(C \setminus \{0\}\). Write down a Laurent series for \(F\).

8. Let \(\Delta\) be the open unit disk and \(\gamma\) the unit circle in \(C\) and \(\phi : \gamma \to C\) a continuous function. Let \(g\) be a meromorphic function on \(C\) that has a single simple pole at 0. Define a function \(f : \Delta \to C\) by the formula:

\[
f(z) = \int_{\gamma} \phi(w) g(w - z) \, dw.
\]

Show that \(f\) is holomorphic.

9. Let \(f\) be a non-constant meromorphic function on \(C\). Show that either there exists a sequence \(z_n \to \infty\) with \(f(z_n) \to 0\) as \(n \to \infty\) or there is a \(z \in C\) such \(f(z) = 0\).

10. Let \(\Omega\) be an open subset of \(C\) and \(f : \Omega \to C\) a holomorphic function. Let \(z\) be a point in \(\Omega\) and assume that \(\Omega\) contains a disk \(D\) centered at \(z\) of radius \(R\). Also assume that \(f(D)\) is contained in a disk \(D'\) centered at \(f(z)\) of radius \(r\). Show that \(|f'(z)| \leq r/R\).